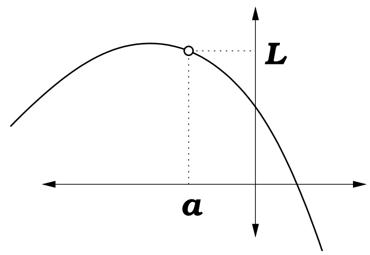
Limits

A limit is a way of defining something by means of successively better and better approximations. Suppose a function $f(x)$ is defined at points x near a but not at $x = a$. Graphically, we have



Now, writing

$$\lim_{x \to a} f(x) = L$$

means that we can make $f(x)$ as close to L as we want by ensuring that x is close enough to a.

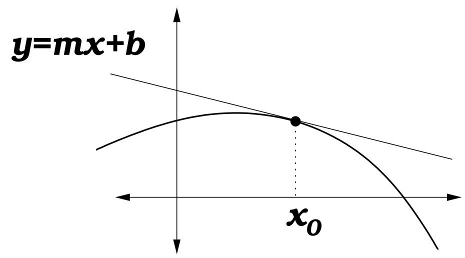
An example of the use of limits is in the definition of the exponential function as the limit of the compound interest formula. Here

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

means that the exponential function can be approximated by taking more and more compounding intervals per time period. Two other examples of using limits to define mathematical functions are the derivative and integral which will be discussed below.

Derivatives

Geometrically, the derivative $f'(x)$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(x, f(x))$. Graphically, we have



where the slope of the tangent line indicated is given by $m = f'(x_0)$. In terms of limits, the slope of the tangent line can be defined by successive approximations using the slopes of secant lines as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

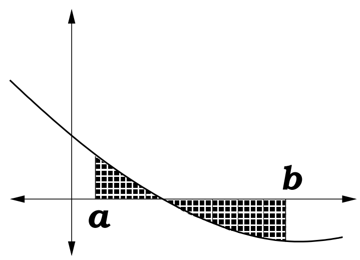
This limit definition can be used to obtain the power rule, the product rule, the quotient rule, the chain rule and other rules of differential calculus.

Integrals

Geometrically, the integral

$$\int_a^b f(x)dx$$

is the area between the curve $y = f(x)$ and the x-axis over the interval $[a, b]$ where the areas below the x-axis are counted as negative. The integral depicted in the graph



is, in this case, negative since the area of the shaded region below the x-axis is larger than the area above the x-axis.

In terms of limits, the integral can be defined by the sums of approximating rectangles with successively narrower and narrower widths. Using the left endpoint to determine the height of the rectangle, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} \left\{ \sum_{k=0}^{n-1} f\Big(a + k\frac{b-a}{n}\Big)\frac{b-a}{n} \right\}.$$

The fundamental theorem of calculus makes a connection between integrals and anti-derivatives which can be used to find many areas.