— Key —

Math 181 Honors Exam 1 Version A

1. Let $A = \{ x \in \mathbf{R} : x > 1 \}$ and $B = \{ x \in \mathbf{R} : x \leq 3 \}$.

   (i) What is $A \cap B$?

   (A) $(1, 3)$
   (B) $[1, 3)$
   (C) $(1, 3]$
   (D) $[1, 3]$
   (E) $(-\infty, \infty)$

   (ii) What is $A \cup B$?

   (A) $(1, 3)$
   (B) $[1, 3)$
   (C) $(1, 3]$
   (D) $[1, 3]$
   (E) $(-\infty, \infty)$

   (iii) What is $A - B$?

   (A) $(3, \infty)$
   (B) $[3, \infty)$
   (C) $(-\infty, 1)$
   (D) $(-\infty, 1]$
   (E) $\emptyset$

2. State Axiom 10: the Least Upper Bound Axiom or Completeness Axiom.

   Any set that is bounded above has a least upper bound in $\mathbb{R}$.

3. Let S be a bounded set and $\alpha = \sup S$. Then

   (A) α is a lower bound for S and if γ is another lower bound then $\alpha \leq \gamma$.
   (B) α is a lower bound for S and if γ is another lower bound then $\alpha \geq \gamma$.
   (C) α is an upper bound for S and if γ is another upper bound then $\alpha \geq \gamma$.
   (D) α is an upper bound for S and if γ is another upper bound then $\alpha \leq \gamma$.
   (E) none of these.

4. Let s be a step function such that $\int_0^4 s(x)\,dx = 7$. Find the following integrals. For full credit explain your reasoning.

(i) $\int_0^4 2s(x)\,dx$ $= 14$,

The function has been stretched vertically by a factor 2, so the integral is $2 \times 7 = 14$.

(ii) $\int_0^2 s(2x)\,dx$ $= \frac{7}{2}$,

The function has been squished horizontally by a factor 2, so the integral is $\frac{7}{2} = 3.5$.

(iii) $\int_0^4 s(4-x)\,dx$ $= 7$,

The function has been reflected about the vertical line $x = 2$. The area stays the same.

5. Complete the square to write the parabola $y = x^2 + 4x + 7$ in the form $y = (x+a)^2 + b$ where $(-a, b)$ is the vertex of the parabola.

$y = (x+2)^2 - 4 + 7 = (x+2)^2 + 3$

The vertex is $(-2, 3)$.

6. Compute the following sums:

(i) $\sum_{k=2}^{11} k = 2+3+\cdots+11 = \frac{(11)(12)}{2} - 1 = 66-1=65$

(ii) $\sum_{k=2}^{11}(2k+1) = 2\sum_{k=2}^{11} k + \sum_{k=2}^{11} 1$

$= 2(65) + 10 = 140$

(iii) $\sum_{k=0}^{3}(k-1)2^k = -1+0+4+2\cdot 2^3$

$= 3+16 = 19$

7. Suppose $x \neq 0$. Find a formula for the sum $\sum_{k=5}^{17}(x^2+1)^k = \sum_{k=5}^{17} y^k$

(A) $\frac{(x^2+1)^{18}-(x^2+1)^5}{x^2}$.

(B) $\frac{(x^2+1)^{17}-(x^2+1)^5}{x^2}$.

(C) $\frac{(x^2+1)^{18}-(x^2+1)^4}{x^2}$.

(D) $\frac{(x^2+1)^{17}-(x^2+1)^4}{x^2}$.

(E) none of these.

$= \frac{y^5-y^{18}}{1-y}$

$= \frac{y^{18}-y^5}{y-1}$

$= \frac{(x^2+1)^{18}-(x^2+1)^5}{x^2+1-1}$

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8. Use induction to prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Let $n=1$. Then $1 = 1^2$ so it is true for $n=1$. Suppose, for induction that it holds for n. Then

$$1+3+5+\cdots+2(n+1)-1 = n^2 + 2(n+1)-1$$
$$= n^2+2n+1 = (n+1)^2$$

implies that $n+1$ holds. This completes the induction and proves the result for all $n \in \mathbb{N}$.

9. Convert the repeating decimal $1.\bar{7}$ into a fraction.

$$1+\frac{7}{10}+\frac{7}{10^2}+\cdots = 1+\frac{7}{10}\left(1+\frac{1}{10}+\cdots\right)$$
$$= 1+\frac{7}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = 1+\frac{7}{10}\left(\frac{10}{9}\right)$$
$$= 1+\frac{7}{9} = \frac{16}{9}$$

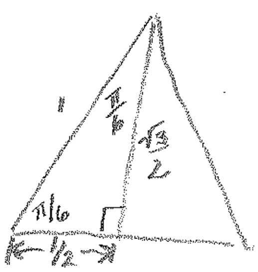
10. Let $[x]$ denote the greatest integer less than x and $\sin x$ and $\cos x$ the sine and cosine of the angle x expressed in radians. Where possible express your answer exactly. Compute the following integrals:

(i) $\displaystyle\int_{-4}^{7/3} [x]\,dx = -4 + (-3) + \cdots + 0 + 1 + 2\left(\frac{1}{3}\right)$

$$= -\frac{4 \cdot 5}{2} + 1 + \frac{2}{3} = -10 + 1 + \frac{2}{3}$$

$$= \frac{-30+5}{3} = \frac{-25}{3}$$

(ii) $\displaystyle\int_{0}^{\pi/6} \cos(x)\,dx = \sin x \Big|_0^{\pi/6} = \sin\frac{\pi}{6} - \sin 0 = \frac{1}{2}$

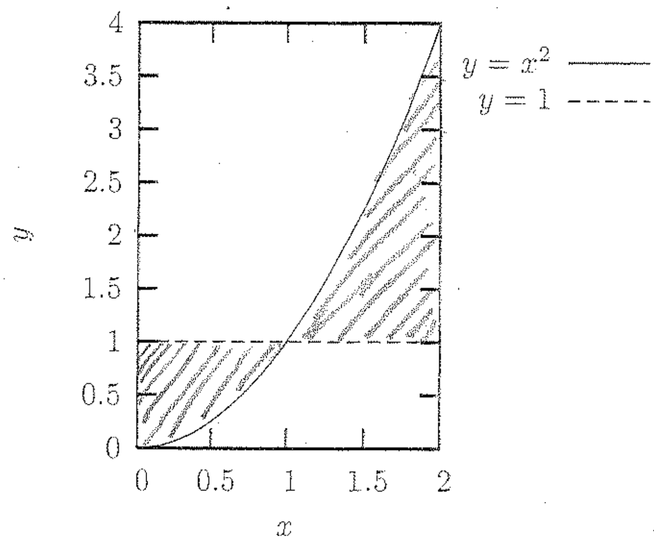


(iii) $\displaystyle\int_{1}^{3} 2x^2\,dx = \frac{2x^3}{3}\Big|_1^3 = \frac{(2)(3^3)}{3} - \frac{2}{3}$

$$= \frac{54-2}{3} = \frac{52}{3}$$

11. Let A$=\int_0^2 |x^2-1|\,dx$.

(i) Color a set of points in the following graph with area equal to A.



(ii) Compute A.

$$A = \int_0^1 (1-x^2)\,dx + \int_1^2 (x^2-1)\,dx$$

$$= -\int_0^1 x^2\,dx + \int_1^2 x^2\,dx$$

$$= -\frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 = -\frac{1}{3} + \frac{8}{3} - \frac{1}{3} = \frac{6}{3} = 2$$

12. Let f be defined as

$$f(x) = \begin{cases} 2 & \text{if } x < 2 \\ 3 & \text{if } x \in [2,4] \\ x & \text{if } x > 4 \end{cases}$$

Find $\int_1^5 f(x)\,dx$. $= \int_1^2 2dx + \int_2^4 3dx + \int_4^5 x\,dx$

$= 2 + 3\cdot 2 + \left(\frac{x^2}{2} \Big|_4^5 \right)$

$= 8 + \frac{25}{2} - \frac{16}{2} = 8 + \frac{9}{2} = \frac{16+9}{2} = \frac{25}{2}$