Math 181 Final Review Sheet A

1. State the following integration and differentiation formula:

$$\int_a^b \sin x\,dx = \boxed{-\cos x\,\Big|_a^b}$$

$$\int_a^b \arcsin x\,dx = \boxed{x\arcsin x + \sqrt{1-x^2}\,\Big|_a^b}$$

assuming $-1 \le a < b \le 1$

$$\int_a^b \cos x\,dx = \boxed{\sin x\,\Big|_a^b}$$

$$\int_a^b \arccos x\,dx = \boxed{x\arccos x - \sqrt{1-x^2}\,\Big|_a^b}$$

assuming $-1 \le a < b \le 1$

$$\int_a^b x^\alpha\,dx = \boxed{\frac{1}{\alpha+1}x^{\alpha+1}\,\Big|_a^b}$$

assuming $\alpha \neq -1$

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\arcsin x\,\Big|_a^b}$$

assuming $-1 < a < b < 1$

$$\int_a^b \ln x\,dx = \boxed{x\ln x - x\,\Big|_a^b}$$

assuming $0 < a < b$

$$\int_a^b \frac{1}{x}\,dx = \boxed{\ln x\,\Big|_a^b}$$

assuming $0 < a < b$

$$\int_a^b \frac{1}{1+x^2}\,dx = \boxed{\arctan x\,\Big|_a^b}$$

$$\int_a^b \arctan x\,dx = \boxed{x\arctan x - \frac{1}{2}\ln(1+x^2)\,\Big|_a^b}$$

$$\int_a^b 5^x\,dx = \boxed{\frac{5^x}{\ln 5}\,\Big|_a^b}$$

$$\frac{d}{dx}\ln x = \boxed{\frac{1}{x}}$$

assuming $x > 0$

$$\frac{d}{dx}\sin x = \boxed{\cos x}$$

$$\frac{d}{dx}\arcsin x = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

assuming $-1 < x < 1$

$$\frac{d}{dx}\cos x = \boxed{-\sin x}$$

$$\frac{d}{dx}\arccos x = \boxed{\frac{-1}{\sqrt{1-x^2}}}$$

assuming $-1 < x < 1$

$$\frac{d}{dx}x^\alpha = \boxed{\alpha x^{\alpha-1}}$$

$$\frac{d}{dx}7^x = \boxed{(\ln 7)7^x}$$

$$\frac{d}{dx}\arctan x = \boxed{\frac{1}{1+x^2}}$$

$$\frac{d}{dx}|x| = \boxed{\frac{x}{|x|}}$$

assuming $x \neq 0$

Show that using induction.

$$1+3+5+7+\cdots(2n-1)=\sum_{k=1}^{n}(2k-1)=n^2$$

Base case $n=1$. Then

$$1=1^2$$

Induction step: Suppose $\sum_{k=1}^{n}(2k-1)=n^2$. Then

$$\sum_{k=1}^{n+1}(2k-1)=\sum_{k=1}^{n}(2k-1)+(2n+1)=n^2+(2n+1)$$

$$=n^2+2n+1=(n+1)^2$$

Show using the limit definition of derivative that $\frac{d}{dx}(f(x)g(x))=\frac{df(x)}{dx}\cdot g(x)+f(x)\cdot\frac{dg(x)}{dx}$.

$$\frac{d}{dx}(f(x)g(x))=\lim_{w\to x}\frac{f(w)g(w)-f(x)g(x)}{w-x}$$

$$=\lim_{w\to x}\frac{f(w)g(w)-f(x)g(w)+f(x)g(w)-f(x)g(x)}{w-x}$$

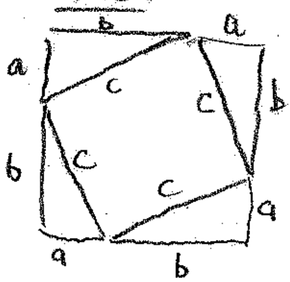
$$=\lim_{w\to x}\frac{f(w)-f(x)}{w-x}\lim_{w\to x}g(w)+f(x)\lim_{w\to x}\frac{g(w)-g(x)}{w-x}$$

$$=f'(x)g(x)+f(x)g'(x)=\frac{df(x)}{dx}\cdot g(x)+f(x)\cdot\frac{dg(x)}{dx}.$$

State and prove the Pythagorean theorem.

Suppose a, b are the lengths of the legs of a right triangle and c is the length of the hypotenuse. Then $a^2+b^2=c^2$.

Proof:



Consider the volume of the figure calculated two ways.

1st as a square with sides $(a+b)$

$$V=(a+b)^2=a^2+2ab+b^2$$

2nd as a square with side c plus four triangles

$$V=c^2+4\cdot\tfrac{1}{2}ab=c^2+2ab$$

Since the volume is the same no matter how calculated then $a^2+b^2=c^2$.

Convert $.0\overline{7}$ into a fraction

$$.0\overline{7}=\sum_{k=2}^{\infty}\frac{7}{10^k}=7\sum_{k=2}^{\infty}\frac{1}{10^k}=7\left(\frac{\frac{1}{10^2}}{1-\frac{1}{10}}\right)$$

$$=7\cdot\frac{1}{10^2}\,\frac{10}{9}=\frac{7}{90}.$$

Find a formula for the sum

$$\sum_{k=7}^{n}(k^2+1)=\sum_{k=1}^{n}k^2-\sum_{k=1}^{6}k^2+(n-6)$$

$$=\frac{n(n+1)(2n+1)}{6}-\frac{6\cdot7\cdot13}{6}+n-6$$

$$=\frac{n(n+1)(2n+1)}{6}+n-97$$

$$\begin{array}{r} 13 \\ \times\ 7 \\ \hline 91 \\ +6 \\ \hline 97 \end{array}$$

Prove using $\delta - \varepsilon$ that $f(x) = \sqrt{x}$ is continuous at $x_0 = 4$.

Let $\varepsilon > 0$ be arbitrary.

Choose $\delta = \min(4, 2\varepsilon)$

Then $|x-4| < \delta$ implies $-4 < x-4 < 4$

so $0 < x < 8$ so $0 < \sqrt{x} < \sqrt{8}$.

Thus

$$|\sqrt{x} - \sqrt{4}| = \left|\sqrt{x} - \sqrt{4}\right| \frac{\sqrt{x}+\sqrt{4}}{\sqrt{x}+\sqrt{4}} = \frac{|x-4|}{\sqrt{x}+\sqrt{4}}$$

$$< \frac{\delta}{\sqrt{x}+\sqrt{4}} \le \frac{\delta}{\sqrt{4}} = \frac{\delta}{2} \le \varepsilon.$$

Find a formula for the sum

$$\sum_{k=n}^{2n} \frac{1}{3^k} = \frac{3}{2}\left(\frac{1}{3^n} - \frac{1}{3^{2n+1}}\right) = \frac{1}{2}\left(\frac{1}{3^{n-1}} - \frac{1}{3^{2n}}\right)$$

Since

$$S = \frac{1}{3^n} + \cancel{\frac{1}{3^{n+1}}} + \cancel{\cdots} + \cancel{\frac{1}{3^{2n}}}$$

$$\frac{1}{3}S = \cancel{\frac{1}{3^{n+1}}} + \cancel{\frac{1}{3^{n+2}}} + \cancel{\cdots} + \frac{1}{3^{2n+1}}$$

$$\left(1 - \frac{1}{3}\right)S = \frac{1}{3^n} - \frac{1}{3^{2n+1}} \quad \text{so} \quad S = \frac{3}{2}\left(\frac{1}{3^n} - \frac{1}{3^{2n+1}}\right)$$

State the Fundamental Theorem of Calculus

Part I Suppose f is continuous on $[a,b]$. Then for x between a and b we have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Part II Suppose $F' = f$ on $[a,b]$. Then for x between a and b we have

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

Show using ε and N and the definition of limit that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ Then $\lim_{n\to\infty}(a_n+b_n) = L+M$.

Let $\varepsilon > 0$ be arbitrary.

Choose $\varepsilon_1 = \varepsilon/2$. Then there is N_1 large enough so that $n \geq N_1$ implies $|a_n - L| < \varepsilon_1$.

Choose $\varepsilon_2 = \varepsilon/2$. Then there is N_2 large enough so that $n \geq N_2$ implies $|b_n - M| < \varepsilon_2$.

Choose $N = \max(N_1, N_2)$. Then $n \geq N$ implies that $n \geq N_1$ and $n \geq N_2$. Therefore

$$|(a_n+b_n)-(L+M)| \leq |a_n - L| + |b_n - M|$$

$$< \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Integrals

$$\int_0^1 \left(x^5 + \ln(x+1)\right)dx = \int_0^1 x^5\,dx + \int_0^1 \ln(x+1)\,dx$$

$$= \frac{x^6}{6}\Big|_0^1 + \int_1^2 \ln x\,dx = \frac{1}{6} + (x\ln x - x)\Big|_1^2$$

$$= \frac{1}{6} + (2\ln 2 - 2) - (1\ln 1 - 1)$$

$$= \frac{1}{6} - 2 + 1 + 2\ln 2 = -\frac{5}{6} + 2\ln 2$$

$$\int_0^1 7^{2x+1}\,dx = \int_0^1 7^{2(x+\frac{1}{2})}\,dx = \int_{1/2}^{3/2} 7^{2x}\,dx$$

$$= \frac{1}{2}\int_1^3 7^x\,dx = \frac{1}{2}\,\frac{7^x}{\ln 7}\Big|_1^3$$

$$= \frac{1}{2\ln 7}\cdot\left(7^3 - 7\right) = \frac{336}{2\ln 7} = \frac{168}{\ln 7}$$

$$\begin{array}{r} 6 \\ \cancel{49} \\ 7 \\ \hline 343 \\ -7 \\ \hline 336 \end{array} \qquad \begin{array}{r} 168 \\ 2\,\overline{)\,336} \\ 136 \\ 16 \end{array}$$

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \frac{1}{2}\int_0^{\sqrt{\pi}} \sin(\varphi(x))\,\varphi'(x)\,dx$$

$$\varphi(x) = x^2 \qquad \varphi'(x) = 2x$$

$$= \frac{1}{2}\int_{\varphi(0)}^{\varphi(\sqrt{\pi})} \sin(x)\,dx = \frac{1}{2}\int_0^{\pi} \sin x\,dx$$

$$= -\frac{1}{2}\cos x\Big|_0^{\pi} = \frac{1}{2}\cos 0 - \frac{1}{2}\cos\pi = 1$$

Derivatives

$$\frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$$

$$\frac{d}{dx}(x\arctan x) = 1\cdot\arctan x + x\cdot\frac{1}{1+x^2}$$

$$= \arctan x + \frac{x}{1+x^2}$$

$$\frac{d}{dx}|\sin x|^3 = 3|\sin x|^2\cdot\frac{d}{dx}|\sin x|$$

$$= 3|\sin x|^2\,\frac{\sin x}{|\sin x|}\cdot\cos x$$

$$= 3|\sin x|\sin x\cdot\cos x$$

$$\frac{d}{dx}\,\frac{x}{x^2+1} = \frac{1\cdot(x^2+1)-x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Find the limits.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2}(x+2) = 2+2 = 4$$

$$\lim_{\theta\to 1} \frac{\theta-1}{\sin(\theta-1)} = \lim_{y\to 0} \frac{y}{\sin y} = \frac{1}{\lim_{y\to 0}\frac{\sin y}{y}} = \frac{1}{1} = 1$$

$y = \theta - 1$ then $y \to 0$ as $\theta \to 1$

$$\lim_{n\to\infty} n(\sqrt[n]{3}-1) = \ln 3$$

using the fact that

$$\ln b = \int_0^b \frac{1}{x}\,dx = \lim_{n\to\infty} n(\sqrt[n]{b}-1).$$

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = e^5$$

using the fact that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \text{ for any } x.$$