Math 181 Quiz 5 Version A

1. Solve the inequality $|2x - 7| < 1$.

$-1 < 2x - 7 < 1$

$6 < 2x < 8$

$3 < x < 4$

Answer:
$(3,4)$

2. Solve the limits

(i) $\lim\limits_{n\to\infty} \dfrac{n+5}{7-2n}$ $= \lim\limits_{n\to\infty} \dfrac{n+5}{7-2n} \cdot \dfrac{\frac{1}{n}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{1+\frac{5}{n}}{\frac{7}{n}-2}$

$= \dfrac{1+0}{0-2} = -\dfrac{1}{2}$.

(ii) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2-3n}\right)$ $\cdot \dfrac{\sqrt{n^2+n}+\sqrt{n^2-3n}}{\sqrt{n^2+n}+\sqrt{n^2-3n}}$

$= \lim\limits_{n\to\infty} \dfrac{(n^2+n)-(n^2-3n)}{\sqrt{n^2+n}+\sqrt{n^2-3n}} = \lim\limits_{n\to\infty} \dfrac{4n}{\sqrt{n^2+n}+\sqrt{n^2-3n}} \cdot \dfrac{\frac{1}{n}}{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \dfrac{4}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{3}{n}}} = \dfrac{4}{\sqrt{1+0}+\sqrt{1-0}} = \dfrac{4}{2} = 2$.

3. Find a formula for $\sum\limits_{k=n}^{2n} (k+1)$

$\sum\limits_{k=n}^{2n} (k+1) = \sum\limits_{k=n+1}^{2n+1} k = \sum\limits_{k=1}^{2n+1} k - \sum\limits_{k=1}^{n} k$

$= \dfrac{(2n+1)(2n+2)}{2} - \dfrac{n(n+1)}{2}$

4. Define

$$e = \lim_{n\to\infty} S_n \qquad \text{where} \qquad S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

In class we showed e was irrational. Provide a proof for one of the following steps:

Step 1. $S_m \le e \le S_m + \frac{1}{m}\frac{1}{m!}$ for every positive m.

Step 2. No fraction $\frac{p}{q}$ could satisfy $S_m \le \frac{p}{q} \le S_m + \frac{1}{m}\frac{1}{m!}$ for every positive m.

Step 1: Set $n > m$. Then

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!} + \frac{1}{(m+1)!} + \cdots + \frac{1}{n!}$$

$$S_m = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!}$$

Therefore

$$S_m \le S_n = S_m + \frac{1}{(m+1)!} + \cdots + \frac{1}{n!}$$

$$= S_m + \frac{1}{(m+1)!}\left(1 + \frac{1}{m+2} + \frac{1}{(m+2)(m+3)} + \cdots + \frac{1}{(m+2)(m+3)\cdots n}\right)$$

$$\le S_m + \frac{1}{(m+1)!}\left(1 + \frac{1}{m+2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{(m+2)^{n-m-1}}\right)$$

$$= S_m + \frac{1}{(m+1)!}\left(\frac{1 - \frac{1}{(m+2)^{n-m}}}{1 - \frac{1}{m+2}}\right)$$

$$\le S_m + \frac{1}{(m+1)!}\,\frac{1}{1 - \frac{1}{m+2}} = S_m + \frac{1}{(m+1)!}\,\frac{m+2}{m+1}$$

$$\le S_m + \frac{1}{(m+1)!} = S_m + \frac{1}{(m+1)}\,\frac{1}{m!}$$

$$\le S_m + \frac{1}{m}\cdot\frac{1}{m!}$$