Key

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1. The Order Axioms are

(POS1) If a, b are positive, so is ab and $a + b$.
(POS2) If a is a number, then either a is positive, or $a = 0$, or $-a$ is positive, and these possibilities are mutually exclusive.

Use the order axioms to show to show that $a > b$ and $b > c$ implies $a > c$.

$a > b$ means $a - b$ is positive
$b > c$ means $b - c$ is positive.
(POS1) implies $(a-b)+(b-c)$ is positive
associativity gives $(a+(-b+b))-c$ is positive
properties of inverses gives $(a+0)-c$ is positive.
properties of identity gives $a-c$ is positive.
Therefore $a > c$.

2. Find all $x \in \mathbf{R}$ such that $\dfrac{1}{x-2} > x$.

$$x - \frac{1}{x-2} = \frac{x^2-2x-1}{x-2} < 0$$

Factor the numerator using the quadratic formula: $a=1$, $b=-2$, $c=-1$

$$r = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

$$\frac{(x-1+\sqrt{2})(x-1-\sqrt{2})}{x-2} < 0$$

	$1-\sqrt{2}$		2		$1+\sqrt{2}$	
$x-1+\sqrt{2}$	$-$		$+$	$+$		$+$
$x-1-\sqrt{2}$	$-$		$-$	$-$		$+$
$x-2$	$-$		$-$	$+$		$+$
	$-$		$+$	$-$		$+$

solution is

$(-\infty, 1-\sqrt{2}) \cup (2, 1+\sqrt{2})$

3. Write the repeating decimal $1.3\overline{7}$ as a fraction.

2
13
9
117

$$1.3\overline{7} = \frac{1}{10}(13.\overline{7}) = \frac{1}{10}\left(13 + \frac{7}{9}\right)$$

$$= \frac{1}{10}\left(\frac{117+7}{9}\right) = \frac{124}{90} = \frac{62}{45}$$

4. Suppose $A = [-1, 7]$ and $B = (2, 3]$.

(i) Find $A \cup B$.

$A \cup B = [-1, 7]$

(ii) Find $A \cap B$.

$A \cap B = (2, 3]$

(iii) Find $A \setminus B$.

$A \setminus B = [-1, 2] \cup (3, 7]$

5. Find the vertex of the parabola $y = 2x^2 + 5x - 1$.

$$2x^2 + 5x - 1 = 2\left(x^2 + \tfrac{5}{2}x\right) - 1$$

$$= 2\left(\left(x + \tfrac{5}{4}\right)^2 - \tfrac{25}{16}\right) - 1$$

$$= 2\left(x + \tfrac{5}{4}\right)^2 - \tfrac{25}{8} - 1$$

$$= 2\left(x + \tfrac{5}{4}\right)^2 - \tfrac{33}{8}$$

$$\begin{array}{r} 1 \\ \frac{25}{8} \\ \hline 33 \end{array}$$

The vertex is $\left(\frac{-5}{4}, \frac{-33}{8}\right)$

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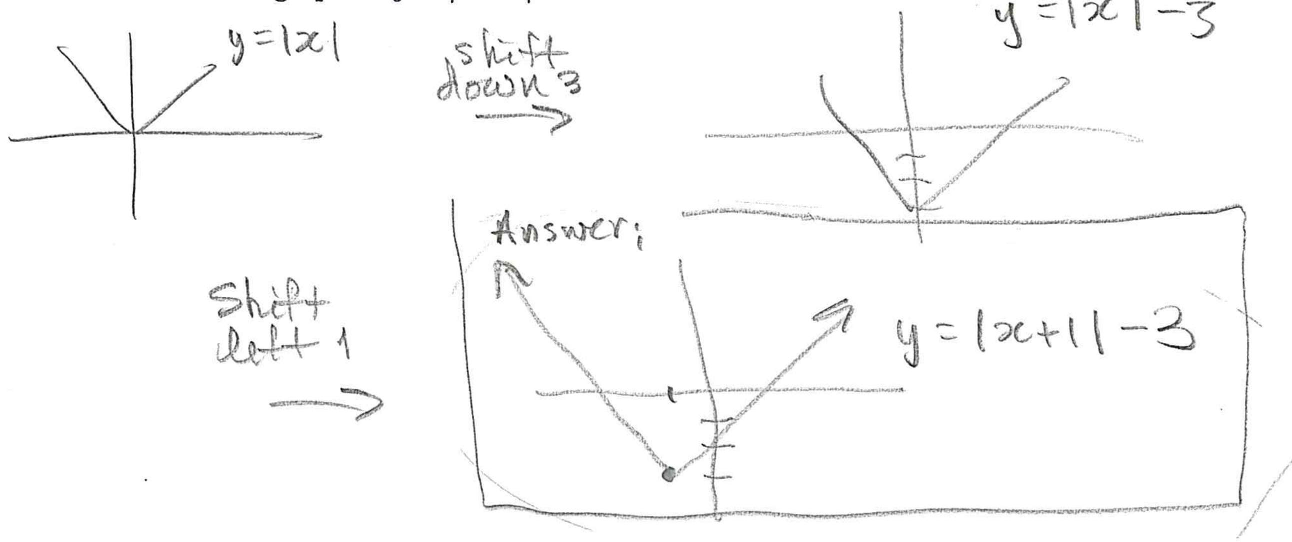
6. Sketch the graph of $y = |x+1| - 3$.



7. Find the domain of the real valued function given by $f(x) = \sqrt{|x+1|-3}$.

Solve: $|x+1|-3 \geq 0$

$|x+1| \geq 3$

So $x+1 \geq 3$ or $x+1 \leq -3$

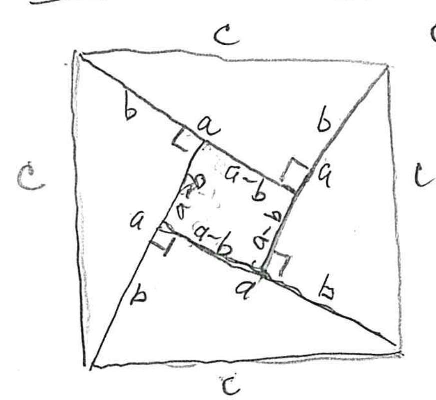
$x \geq 2$ or $x \leq -4$

Answer: domain is $(-\infty, -4] \cup [2, \infty)$

8. State the hypothesis and conclusion and then prove the Pythagorean theorem.

Given a right triangle with legs of lengths a and b and hypotenuse of length c. Then $c^2 = a^2 + b^2$.

Proof: Consider the figure made by arranging 4 copies of the triangle in a c by c square as indicated. Compute the area of the big square in two ways:



$A_1 = c^2$

and

$A_2 = 4$ triangles + little square

$= 4(\frac{1}{2}ab) + (a-b)^2$

$= 2ab + a^2 - 2ab + b^2 = a^2 + b^2$

Since $A_1 = A_2$ then $c^2 = a^2 + b^2$.

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9. Write the continued fraction $[1, \overline{4}]$ in the form $\dfrac{a+\sqrt{b}}{c}$.

$$1 + \boxed{\frac{1}{4+\frac{1}{4+\ldots}}} = 1 + \frac{1}{4+\boxed{\frac{1}{4+\ldots}}}$$

Thus $\cancel{1} + x = \cancel{1} + \dfrac{1}{4+x}$

$$x^2 + 4x - 1 = 0$$

$a = 1 \quad b = 4 \quad c = -1$

$$x = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}$$

Therefore

$$[1, \overline{4}] = 1 + x = -1 + \sqrt{5}$$

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10. State the meaning of $\lim_{x \to a} f(x) = L$ in terms of ϵ and δ.

For every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in \text{domain} f \setminus \{a\}$ and $|x-a| < \delta$ implies $|f(x) - L| < \varepsilon$.

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11. Use the ϵ-δ definition to verify $\lim_{x \to 2} x^2 = 4$.

Let $\varepsilon > 0$ and choose $\delta = \min(1, \varepsilon/5)$

Then $x \in \mathbb{R} \setminus \{2\}$ and $|x-2| < \delta$ implies

$|x-2| < 1$ so $-1 < x-2 < 1$ so $1 < x < 3$.

Therefore

$$|x^2 - 4| = |x^2 - 2x + 2x - 4| \le |x^2 - 2x| + |2x - 4|$$
$$= |x||x-2| + 2|x-2|$$
$$< 3\delta + 2\delta = 5\delta \le 5\frac{\varepsilon}{5} = \varepsilon.$$

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12. The 6 limit laws are

(0) $\lim_{x\to a} c = c$

(1) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

(2) $\lim_{x\to a} \big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(3) $\lim_{x\to a} \big(f(x)g(x)\big) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

(4) $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)}$ provided $\lim_{x\to a} f(x) \neq 0$

(5) $\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$ if f is continuous at $\lim_{x\to a} g(x)$.

(i) Use the ϵ-δ definition to verify limit law 2.

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$ and let $\varepsilon > 0$.
Choose $\varepsilon_1 = \varepsilon/2$. By hypothesis there is $\delta_1 > 0$ such that $x \in \text{domain} f \setminus \{a\}$ and $|x-a| < \delta_1$ implies $|f(x)-L| < \varepsilon_1$.
Choose $\varepsilon_2 = \varepsilon/2$. By " $\delta_2 > 0$ such that $x \in \text{domain}\, g \setminus \{a\}$ and $|x-a| < \delta_2$ implies $|g(x)-K| < \varepsilon_2$.
Choose $\delta = \min(\delta_1, \delta_2)$. Then $x \in \text{domain}(f+g)\setminus\{a\}$ and $|x-a| < \delta$ implies $x \in \text{domain} f \setminus \{a\}$ and $|x-a| < \delta_1$ so that $|f(x)-L| < \varepsilon_1$ and $x \in \text{domain}\, g\setminus\{a\}$ and $|x-a| < \delta_1$ so that $|g(x)-K| < \varepsilon_2$.
Therefore

$$|(f(x)+g(x))-(L+K)| \leq |f(x)-L| + |g(x)-K| < \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence

$$\lim_{x\to a} (f(x)+g(x)) = L+K = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

(ii) Use the limit laws (and the fact that $\lim_{x\to 2} x = 2$) to show $f(x) = \frac{1}{x+1}$ is continuous at the point $x = 2$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{1}{x+1} \overset{(LL4)}{=} \frac{1}{\lim_{x\to 2} x+1}$$

$$\overset{(LL2)}{=} \frac{1}{\lim_{x\to 2} x + \lim_{x\to 2} 1} \overset{(LL0)}{=} \frac{1}{2+1}$$

$$= \frac{1}{3} = f(2).$$

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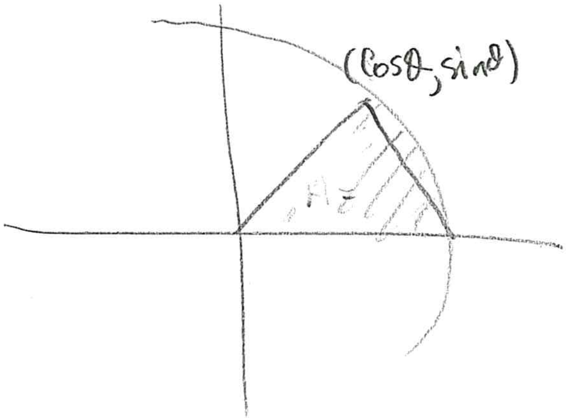
13. [Extra Credit] Use geometry to show $\lim_{x\to 0} \sin x = 0$



for $x \in [0, \frac{\pi}{2}]$ the area of the triangle is contained in the pie slice.

Thus

$$A_{\text{Triangle}} = \frac{1}{2}(1 \cdot \sin x)$$

$$A_{\text{PIE SLICE}} = \frac{x}{2}$$

Therefore $\sin x \leq x$ for $x \in [0, \frac{\pi}{2}]$.

If $x \in [-\frac{\pi}{2}, 0]$. The fact that $\sin x$ is an odd function yields that

$$|\sin x| = \sin|x| \leq |x|$$

Therefore $|\sin x| \leq |x|$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now let $\varepsilon > 0$, Choose $\delta = \min(\frac{\pi}{2}, \varepsilon)$

Then $x \in \mathbb{R} \setminus \{0\}$ and $|x - 0| < \delta$ implies $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ so

$$|\sin x - 0| = |\sin x| \leq |x| < \delta \leq \varepsilon$$

It follows that $\lim_{x \to 0} \sin x = 0$.