$$\frac{13}{9} = 1 + \frac{4}{9}$$

$$= 1 + \frac{1}{9/4}$$

$$\boxed{= 1 + \cfrac{1}{2 + \cfrac{1}{4}}}$$

We have already seen that $13/9$ could be written as $1.\overline{4}$. Using some algebraic manipulation, we can also write it as a <u>continued fraction</u>.

To simplify this, we can write:

$$\frac{13}{9} = [1, 2, 4]$$

The numbers in brackets give the integer part of each denominator.
Other examples:

$$[1, 2, 3, 4] = 1 + \cfrac{1}{2 + \cfrac{1}{3 + 1/4}} = \frac{43}{30} = 1.4\overline{3}$$

$$[3, 5, 2] = 3 + \cfrac{1}{5 + \frac{1}{2}} = \frac{35}{11} = 3.\overline{18}$$

With only a finite number of terms, we could use basic arithmetic to simplify any continued fraction to "simple" fraction or rational number! This is similar to the way that we can simplify decimals with a finite number of digits after the decimal. Now recall that we were able to make sense of decimals with an infinite number of digits after the decimal as long as the digits repeated in periodic pattern. For instance

$$0.\overline{54} = \frac{54}{99} = \frac{6}{11} \quad \text{or} \quad 1.\overline{2} = 1 + \frac{2}{9} = \frac{11}{9}$$

Can we similarly make sense of an infinite, periodic continued fraction? For instance, does $[1, 2, 2, 2, 2, \ldots]$ represent a recognizable number? We could try estimating this number by ignoring the trailing terms:

$$\left.\begin{array}{ll}
[1] & = 1.0 \\
[1, 2] & = 1.5 \\
[1, 2, 2] & = 1.4 \\
[1, 2, 2, 2] & = 1.41\overline{6} \\
[1, 2, 2, 2, 2] & \approx 1.4138 \\
[1, 2, 2, 2, 2, 2] & \approx 1.4143 \\
[1, 2, 2, 2, 2, 2, 2] & \approx 1.4142
\end{array}\right\} \text{Are these getting closer to some value?}$$

These values seem to be getting nearer to something like 1.414.

Let's try something a bit more analytical: we will call our mystery number α:

$$\alpha = [1, 2, 2, 2, \ldots]$$

Expanding this out, we have

$$\alpha = 1 + \underbrace{\boxed{\cfrac{1}{2 + \boxed{\cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}}}_{\parallel \atop x}$$

The quantities in the two boxes are the same. That is, each box contains

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

Let's call this quantity x.

$$\alpha = \underbrace{1 + x = 1 + \frac{1}{2 + x}}$$

we can solve this:

$$1 + x = 1 + \frac{1}{2+x}$$

$$x = \frac{1}{2+x}$$

$$x(2+x) = 1$$

$$x^2 + 2x - 1 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2} = -1 \pm \sqrt{2}$$

Thus either

$$\alpha = 1 + (-1 + \sqrt{2}) = \sqrt{2} \qquad \text{or} \qquad \alpha = 1 + (-1 - \sqrt{2}) = -\sqrt{2}$$

We know that α is positive, thus $\alpha = \sqrt{2}$.

Thus we can make sense of an infinite periodic continued fraction — such continued fractions are <u>square roots</u>! Periodic decimals are rational numbers, and periodic continued fractions are roots, which may not be rational!

How close is a truncated continued fraction to the number it represents? Consider $\varphi = [1, 1, 1, 1, \ldots]$:

<u>1 term:</u>

$$\varphi \approx 1 + \frac{1}{\underbrace{\text{stuff}}_{\in(0,1)}} \in (1, 2)$$

off by at most 1
(decimal approx off by 1)

<u>2 terms:</u>

$$\varphi \approx 1 + \underbrace{\frac{1}{1 + \underbrace{\frac{1}{\text{stuff}}}_{\in(0,1)}}}_{\in(\frac{1}{2}, 1)} \in \left(\frac{3}{2}, 2\right)$$

off by at most 1/2
(decimal approx off by 1/10)

<u>3 terms:</u>

$$\varphi \approx 1 + \underbrace{\frac{1}{1 + \underbrace{\frac{1}{1 + \underbrace{\frac{1}{\text{stuff}}}_{\in(0,1)}}}_{\in(\frac{1}{2}, 1)}}}_{\in(\frac{1}{2}, \frac{2}{3})} \in \left(\frac{3}{2}, \frac{5}{3}\right)$$

off by at most 1/6
(decimal approx off by 1/100)

<u>4 terms:</u>

$$\varphi \approx 1 + \underbrace{\frac{1}{1 + \underbrace{\frac{1}{1 + \frac{1}{1 + \frac{1}{\text{stuff}}}}}_{\in(\frac{1}{2}, \frac{2}{3})}}}_{\in(\frac{3}{5}, \frac{2}{3})} \in \left(\frac{8}{5}, \frac{5}{3}\right)$$

off by at most 1/15
(decimal approx off by 1/1000)

The error is getting smaller & smaller. We won't <u>prove</u> this, but the approximations do converge to a limit. The rate of convergence depends <u>a lot</u> on the size of the terms in the continued fraction - bigger terms converge much faster.

Some interesting examples:

$$\varphi = [1, 1, 1, 1, \ldots] \approx 1.618$$

φ is often called the golden ratio. The truncated continued fraction representations of φ converge very slowly compared to other numbers — in fact, φ is, in some sense, the hardest number to approximate with fractions. φ is closely related to the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots] \approx 2.718$$

e is the base of natural logarithms, and will come up often in this class. Its decimal representation has no pattern, but the continued fraction representation is quite elegant.

$$\pi \approx [3, 7, 15, 1, 292, 1, 1, \ldots] \approx 3.142$$

π is the ratio of a circle's circumference to its diameter. The continued fraction representation, like the decimal representation, has no pattern. However, $[3, 7, 15, 1] = 355/113$ is a really good approximation – so is the easy to remember $[3, 7] = 22/7$.