27 SEPT HOMEWORK #5

① (i) $|x+1| < 2x \Rightarrow x+1 < 2x$ and $x > 0$ *

$1 < x$

i.e. $x > 1$

* Note that $2x$ is greater than the absolute value of something, thus x must be greater than 0.

$x \in (1, \infty)$.

① (ii) $|x^2 - 5| \geq 1 \Rightarrow$ Interval Endpoints:

$|x^2-5| = 1 \Rightarrow x^2 - 5 = \pm 1$

$x^2 = 5 \pm 1$

$x = \pm\sqrt{5 \pm 1}$

$= \pm\sqrt{6}, \pm 2$

$(-\infty, -\sqrt{6})$: e.g. $x = -3 \Rightarrow |(-3)^2 - 5| = |9-5| = 4 \geq 1$ ✓

$(-\sqrt{6}, -2)$: e.g. $x = -7/3 \Rightarrow |(-7/3)^2 - 5| = |49/9 - 5| = 5/9 \not\geq 1$ X

$(-2, 2)$: e.g. $x = 0 \Rightarrow |0^2 - 5| = |-5| = 5 \geq 1$ ✓

$(2, \sqrt{6})$: e.g. $x = 7/3 \Rightarrow$ same as above X

$(\sqrt{6}, \infty)$: e.g. $x = 3 \Rightarrow$ same as above ✓

$$\boxed{x \in (-\infty, -\sqrt{6}] \cup [-2, 2] \cup [\sqrt{6}, \infty)}$$

① (iii) $x + \frac{1}{x} \leq 7 \Rightarrow x - 7 + \frac{1}{x} \leq 0 \Rightarrow \frac{x^2 - 7x + 1}{x} \leq 0$

either

$x^2 - 7x + 1 \geq 0$

$x < 0$

or

$x^2 - 7x + 1 \leq 0$

$x > 0$

$$x^2 - 7x + 1 = 0 \Rightarrow x = \frac{7 \pm \sqrt{49 - 4(1)(1)}}{2} = \frac{7 \pm \sqrt{45}}{2} = \frac{7 \pm 3\sqrt{5}}{2}$$

$(-\infty, 0)$: e.g. $x = -1 \Rightarrow -1 + \frac{1}{-1} = -2 \leq 7$ ✓

$(0, 7-3\sqrt{5}/2)$: e.g. $x = 7-3\sqrt{5}/4 \Rightarrow 7-3\sqrt{5}/4 + 4/7-3\sqrt{5} \approx 13.7812 \not\leq 7$ X

$(7-3\sqrt{5}/2, 7+3\sqrt{5}/2)$: e.g. $x = 1 \Rightarrow 1 + \frac{1}{1} = 2 \leq 7$ ✓

$(7+3\sqrt{5}/2, \infty)$: e.g. $x = 10 \Rightarrow 10 + \frac{1}{10} = 10.1 \not\leq 7$ X

$$\boxed{x \in (-\infty, 0] \cup \left[\frac{7-5\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2}\right]}$$

② (i) Let $\varepsilon > 0$ and choose $\delta < \min\{3/2, \varepsilon\}$. Suppose that $x \in \mathbb{R} \setminus \{-1, 1\}$ and that $|x-1| < \delta$. As $|x-1| < \delta$ and $\delta < 3/2$, we have

$$-3/2 < x-1 < 3/2 \implies -3 < 2x-2 < 3$$
$$\implies 1 < 2x+2 < 7.$$

In particular, $|2x+2| = 2x+2 > 1 > 0$. Then

$$\left|\frac{x}{x+1} - \frac{1}{2}\right| = \left|\frac{2x-x-1}{2x+2}\right|$$

$$= \frac{|x-1|}{|2x+2|}$$

$$< \frac{\delta}{|2x+2|} \qquad \text{(by assumption)}$$

$$< \delta \qquad (\text{as } |2x+2| > 1)$$

$$< \varepsilon.$$

∎

② (ii) Let $\varepsilon > 0$ and choose $\delta < \min\{1, \varepsilon\}$. Suppose that $x \in \mathbb{R} \setminus \{-3, 0\}$, and that $|x+3| < \delta$. As $\delta < 1$, we have

$$-1 < x+3 < 1 \implies -4 < x < -2$$

and so $x^2 \in (4, 16)$. Moreover,

$$-1 < x+3 < 1 \implies -7 < x-3 < -5,$$

which implies that $|x-3| < 7$. Combining these, we have

$$\frac{|x-3|}{9x^2} < \frac{7}{9 \cdot 4} = \frac{7}{36} < 1.$$

Then

$$\left|\frac{1}{x^2} - \frac{1}{9}\right| = \left|\frac{9-x^2}{9x^2}\right| = \frac{|x+3|\,|x-3|}{9x^2} < \delta\,\frac{|x-3|}{9x^2} < \delta < \varepsilon.$$

by the above result

∎

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(2)(iii) Let $\varepsilon > 0$ and choose $\delta < \min\{1, \varepsilon/19\}$. Suppose that $x \in \mathbb{R} \setminus \{2\}$ and that $|x-2| < \delta$. as $\delta < 1$, we have

$$-1 < x-2 < 1 \Rightarrow 1 < x < 3.$$

In particular, $x < 3$ and so $x^2 + 2x + 4 < 19$. Then

$$|x^3 - 8| = |(x-2)(x^2+2x+4)| < \delta \cdot 19 < \frac{\varepsilon}{19} \cdot 19 < \varepsilon.$$

∎

(3)(i) Let $\varepsilon > 0$ and choose δ so that if $|x-4| < \delta$ then $|f(x) - 5| < \varepsilon/3$. Suppose that $x \in \mathcal{D}(f) \setminus \{4\}$ and that $|x-4| < \delta$. Then

$$|3 \cdot f(x) - 15| = 3|f(x) - 15| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

∎

(3)(ii) Let $\varepsilon > 0$ and choose $\delta < \frac{\varepsilon}{2}$ such that if $|x-4| < \delta$ then $|g(x) + 2| < \frac{\varepsilon}{2}$. Th

$$\begin{aligned} |x + g(x) - 2| &= |g(x) + 2 + x - 4| \\ &\le |g(x)+2| + |x-4| \quad \text{triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

∎

(3)(iii) Choose δ_1 such that if $|y-4| < \delta_1$, then $|f(y) - 5| < \varepsilon$. Set $\delta = \delta_1/4$. Th if $|x-1| < \delta$, we have

$$|4x - 4| < 4\delta = \delta_1.$$

Set $y = 4x$. As $|y-4| < \delta_1$, we have that $|f(y) - 5| < \varepsilon$. Substituting back in, $|f(4x) - 5| < \varepsilon$, and we are done.

∎

④ (i)

$x_1 = 2.0000000$
$x_2 \approx 3.4142136$
$x_3 \approx 3.8477591$
$x_4 \approx 3.9615706$
$x_5 \approx 3.9903695$

④ (ii)

$$\underbrace{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}_{\text{call the whole thing } x} = 2+\sqrt{x} = x$$

Solving for x, we have

$$2+\sqrt{x} = x \implies \sqrt{x} = x-2$$
$$\implies x = (x-2)^2 = x^2-4x+4$$
$$\implies x^2-5x+4 = 0$$
$$\implies (x-4)(x-1) = 0$$
$$\implies x = 4 \quad \text{or} \quad x = 1$$

Clearly $x \geq 1$, so $2+\sqrt{x} > 2$. But then x cannot be 1. Therefore $\boxed{x = 4.}$