– Key –

Math 181 Midterm Version A

1. Precisely define $\lim_{x\to a^+} f(x) = L$ using inequalities in terms of δ and ϵ.

For every $\varepsilon > 0$ there is $\delta > 0$ such that if $a < x < a+\delta$ then $|f(x) - L| < \varepsilon$.

2. Find the following limits:

(i) $\lim_{x\to 0} \cos x$ $= \cos 0 = 1$,

since cosine is continuous.

(ii) $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$ $= \lim_{x\to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$

$= \lim_{x\to 3} (x+3) = 3+3 = 6$

(iii) $\lim_{x\to\infty} \dfrac{x^2+x-3}{2x^2-4}$ $\cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{1+\frac{1}{x}-\frac{3}{x^2}}{2-\frac{4}{x^2}}$

$= \dfrac{1+0-0}{2-0} = \dfrac{1}{2}$

3. Define the derivative $f'(x)$ of a function $f(x)$ using limits.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

4. Use the limit definition to explain why the derivative of $f(x) = 1/x$ is $f'(x) = -1/x^2$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{x - (x+h)}{h(x+h)x}$$

$$= \frac{-h}{\cancel{h}(x+h)x} = \frac{-1}{(x+h)x}$$

Therefore

$$f'(x) = \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}.$$

5. Answer the following true/false questions:

(i) If f is differentiable at a, then f is continuous at a.

(A) True

(B) False

(ii) e is the number such that $\lim_{h \to 0} \dfrac{e^h + 1}{h} = 1$.

(A) True

(B) False

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6. State the following derivative rules from memory:

$$\frac{d}{dx}\sin x = \boxed{\cos x} \qquad \frac{d}{dx}(fg)(x) = \boxed{f'(x)g(x) + f(x)g'(x)}$$

$$\frac{d}{dx}\cos x = \boxed{-\sin x} \qquad \frac{d}{dx}(f \circ g)(x) = \boxed{f'(g(x))\, g'(x)}$$

$$\frac{d}{dx}\tan x = \boxed{\sec^2 x} \qquad \frac{d}{dx}\left(\frac{f}{g}\right)(x) = \boxed{\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$$

$$\frac{d}{dx}x^\alpha = \boxed{\alpha x^{\alpha-1}} \qquad \frac{d}{dx}\ln x = \boxed{\frac{1}{x}} \qquad \frac{d}{dx}\log_b x = \boxed{\frac{1}{x\ln b}}$$

$$\frac{d}{dx}\arccos x = \boxed{\frac{-1}{\sqrt{1-x^2}}} \qquad \frac{d}{dx}\sec x = \boxed{\sec x \tan x} \qquad \frac{d}{dx}e^x = \boxed{e^x}$$

$$\frac{d}{dx}\arcsin x = \boxed{\frac{1}{\sqrt{1-x^2}}} \qquad \frac{d}{dx}\csc x = \boxed{-\csc x \cot x} \qquad \frac{d}{dx}a^x = \boxed{a^x \ln a}$$

$$\frac{d}{dx}\arctan x = \boxed{\frac{1}{1+x^2}} \qquad \frac{d}{dx}\cot x = \boxed{-\csc^2 x} \qquad \frac{d}{dx}|x| = \boxed{\frac{x}{|x|}}$$

7. Use the rules of calculus to compute the following derivatives:

(i) $\frac{d}{dx}(x \sin x)$ $= 1 \cdot \sin x + x \cos x$

by prod. rule.

(ii) $\frac{d}{dx} \arctan(1+x^2)$ $= \frac{1}{1+(1+x^2)^2} \cdot \frac{d}{dx}(1+x^2)$

$= \frac{1}{1+(1+x^2)^2} \cdot 2x$

(iii) $\frac{d}{dx}\left(\frac{x^3-5}{x^2+4}\right)$ $= \frac{3x^2(x^2+4)-(x^3-5)2x}{(x^2+4)^2}$

by quotient rule.

(iv) $\frac{d}{dx} x^x$ $= \frac{d}{dx} e^{x \ln x} = e^{x \ln x} \frac{d}{dx}(x \ln x)$

$= x^x \left[\ln x + x \cdot \frac{1}{x} \right] = x^x (1 + \ln x)$

8. Consider the curve defined by the equation $x^3 + y^3 = 6xy$.

(i) Use implicit differentiation to find y' in terms of x and y.

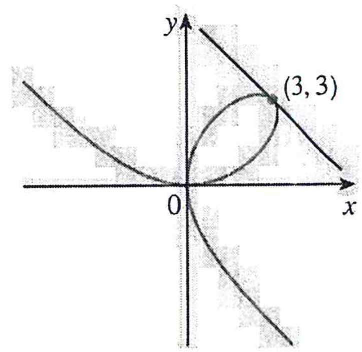
$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$

$$3x^2 + 3y^2y' = 6y + 6xy'$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

(ii) Find equation of the line tangent to this curve at the point $(3, 3)$.

$$\text{Slope} = y'\Big|_{(x,y)=(3,3)} = \frac{6\cdot 3 - 3\cdot 3^2}{3\cdot 3^2 - 6\cdot 3} = -1$$



$$y - 3 = (-1)(x-3)$$

$$y = -x + 6$$

(iii) At what point in the first quadrant is the tangent line horizontal?

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = 0$$

$$6y = 3x^2$$

$$y = \frac{1}{2}x^2$$

plug into equation...

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$

$$x^3 + \frac{1}{8}x^6 = 3x^3$$

$$\frac{1}{8}x^6 = 2x^3$$

$$x^3 = 16$$

$$x = \sqrt[3]{16}$$

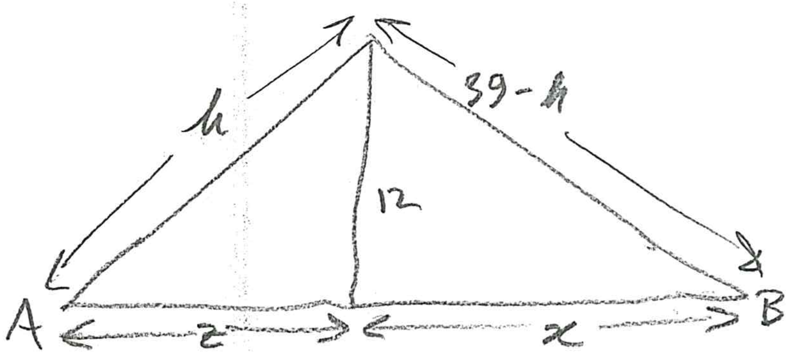
plug in to find y...

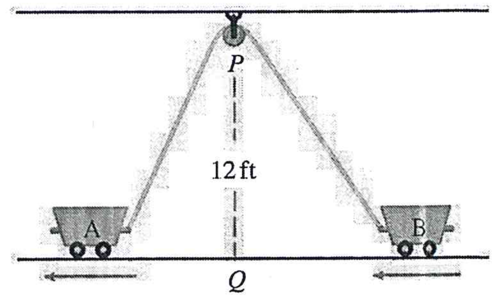
$$y = \frac{1}{2}\left(\sqrt[3]{16}\right)^2 = \frac{1}{2}(16)^{2/3}$$

The point is

$$(x,y) = \left(\sqrt[3]{16}, \frac{1}{2}(16)^{2/3}\right)$$

9. Two carts, A and B, are connected by a rope 39 ft long that passes over a pully P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?





Pythagorean theorem

$$z^2 + 12^2 = h^2 \quad \text{and} \quad x^2 + 12^2 = (39-h)^2$$

Differentiating

$$2z\frac{dz}{dt} = 2h\frac{dh}{dt} \quad \text{and} \quad 2x\frac{dx}{dt} = 2(39-h)(-1)\frac{dh}{dt}$$

Data:

$$z = 5, \quad \frac{dz}{dt} = 2$$

Solving

$$h = \sqrt{12^2 + z^2} = \sqrt{12^2 + 5^2} = 13$$

$$x = \sqrt{(39-h)^2 - 12^2} = \sqrt{26^2 - 12^2} = \sqrt{532} = 2\sqrt{133}$$

$$\frac{dh}{dt} = \frac{z}{h}\frac{dz}{dt} = \frac{5}{13}\cdot 2 = \frac{10}{13}$$

$$\frac{dx}{dt} = \frac{h-39}{x}\cdot\frac{dh}{dt} = \frac{-26}{2\sqrt{133}}\cdot\frac{10}{13} = \frac{-130}{13\sqrt{133}} \approx -.8671$$

The cart B is moving at approx. 0.8671 ft/sec toward point Q.