Math 182 Honors Quiz 5 ~ Key ~

1. Taylor's theorem states

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos c$$

for some c between 0 and x.

(i) Let r be the positive solution to $x = \cos x$. Use the quadratic Taylor polynomial approximation of $\cos x$ to approximate r. Let x be the approximation

$1 - \frac{x^2}{2} = x$, $\quad x^2 + 2x - 2 = 0$

$x = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$

r is approximately $-1 + \sqrt{3}$

(ii) Use the remainder term in Taylor's formula to find a bound on $|x - \cos x|$. where x is the approximation from part (i).

$$|x - \cos x| = \left| x - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \cos c\right)\right|$$

$$= \left|\frac{x^4}{4!} \cos c\right| \leq \left|\frac{x^4}{4!}\right| = \frac{(-1+\sqrt{3})^4}{4!}$$

$$= \frac{7}{6} - \frac{2}{\sqrt{3}} < 0.012$$

2. Find the function $y = y(x)$ that satisfies the ordinary differential equation initial value problem

$$\begin{cases} y' + 3y = x \\ y(0) = 2. \end{cases}$$

$$\frac{d}{dx}(ye^{3x}) = xe^{3x}$$

$$ye^{3x} = \int xe^{3x}\,dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}\,dx = \left(\frac{x}{3} - \frac{1}{9}\right)e^{3x} + C$$

$$f(x) = x \qquad f'(x) = 1$$
$$g'(x) = e^{3x} \qquad g(x) = \frac{1}{3}e^{3x}$$

Therefore

$$y = Ce^{-3x} + \frac{x}{3} - \frac{1}{9} \qquad y(0) = 2 = C - \frac{1}{9} \qquad C = \frac{19}{9}$$

and the solution is

$$y = \frac{19}{9}e^{-3x} + \frac{x}{3} - \frac{1}{3}$$

3. Find the limit $\lim_{x\to 0} \frac{1-\cos(3x)}{\sin(x^2)} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{(3x)^2}{2} + o(x^3)\right)}{x^2 + o(x^4)}$

$$= \lim_{x\to 0} \frac{\frac{9}{2}x^2 + o(x^3)}{x^2 + o(x^4)} = \lim_{x\to 0} \frac{\frac{9}{2} + o(x)}{1 + o(x^2)}$$

$$= \frac{9}{2}$$

4. Find the antiderivative $\int \frac{dx}{2x^2+3x+4} = \frac{1}{2}\int \frac{dx}{x^2 + \frac{3}{2}x + 2} = \frac{1}{2}\int \frac{dx}{(x+\frac{3}{4})^2 + 2 - \frac{9}{16}}$

$$\begin{array}{r} 32 \\ -9 \\ \hline 23 \end{array}$$

$$= \frac{1}{2}\cdot\frac{16}{23}\int \frac{dx}{(x+\frac{3}{4})^2\frac{16}{23} + 1} \qquad \text{let } u = \frac{4}{\sqrt{23}}\left(x+\frac{3}{4}\right)$$

$$du = \frac{4}{\sqrt{23}}\,dx$$

$$= \frac{8}{23}\,\frac{\sqrt{23}}{4}\int \frac{du}{u^2+1} = \frac{2}{\sqrt{23}}\arctan u + C$$

$$= \frac{2}{\sqrt{23}}\arctan\left(\frac{4}{\sqrt{23}}\left(x+\frac{3}{4}\right)\right) + C$$