1.)

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \text{constant}$$

Possible intermediate steps:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

For the integrand $\frac{x}{\sqrt{1-x^2}}$, substitute $u = 1 - x^2$ and $du = -2\,x\,dx$:

$$= -\frac{1}{2} \int \frac{1}{\sqrt{u}}\,du$$

The integral of $\frac{1}{\sqrt{u}}$ is $2\sqrt{u}$:

$$= -\sqrt{u} + \text{constant}$$

Substitute back for $u = 1 - x^2$:

$$= -\sqrt{1-x^2} + \text{constant}$$

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2.

Possible intermediate steps:

$$\int x^4 \log(x)\, dx$$

For the integrand $x^4 \log(x)$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where
$f = \log(x), \quad dg = x^4 dx,$

$df = \frac{1}{x}dx, \quad g = \frac{x^5}{5}$:

$$= \frac{1}{5} x^5 \log(x) - \frac{1}{5} \int x^4\, dx$$

The integral of x^4 is $\frac{x^5}{5}$:

$$= \frac{1}{5} x^5 \log(x) - \frac{x^5}{25} + \text{constant}$$

(3) $\int \sin^2 x \cos^5 x \, dx$

$\int \sin^2 x \cos x (1-\sin^2 x)^2 \, dx$

• convert $(\cos^2 x)$ to $(1-\sin^2 x)$

$\int \sin^2 x \cos x (1 - 2\sin^2 x + \sin^4 x) \, dx$

• expand $(1-\sin^2 x)^2$

$\int \cos x (\sin^2 x - 2\sin^4 x + \sin^6 x) \, dx$

• multiply out $\sin^2 x$

$u = \sin x$

$du = \cos x \, dx$

$\int (u^2 - 2u^4 + u^6) \, du$

• use a u-substitution

$\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$

• integrate

$\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

• put in terms of x

DC

Problem 4

$$\int \frac{1}{x^3 + 4x} dx$$

Begin by transforming the main integral using partial fractions

$$\int (\frac{1}{4x} - \frac{x}{4(x^2 + 4)}) dx$$

Split the integral up into two smaller integrals

$$\frac{1}{4}\int \frac{1}{x} dx - \frac{1}{4}\int \frac{x}{x^2 + 4} dx$$

For the second integral, make the substitution u = x^2 + 4 and du = 2xdx

$$\frac{1}{4}\int \frac{1}{x} dx - \frac{1}{8}\int \frac{1}{u} du$$

The integral of $\frac{1}{x}$ is ln(x), and the integral of $\frac{1}{u}$ is ln(u)

$$\frac{\ln(x)}{4} - \frac{\ln(u)}{8} + C$$

Substitute x^2 + 4 back in for u

$$\frac{\ln(x)}{4} - \frac{\ln(x^2 + 4)}{8} + C$$

5

Possible intermediate steps:

$$\int \frac{\sqrt{1+\log(x)}}{x\log(x)}\,dx$$

For the integrand $\frac{\sqrt{\log(x)+1}}{x\log(x)}$, substitute $u=\log(x)$ and $du=\frac{1}{x}dx$:

$$=\int \frac{\sqrt{u+1}}{u}\,du$$

For the integrand $\frac{\sqrt{u+1}}{u}$, substitute $s=\sqrt{u+1}$ and $ds=\frac{1}{2\sqrt{u+1}}du$:

$$=2\int \frac{s^2}{s^2-1}\,ds$$

For the integrand $\frac{s^2}{s^2-1}$, do long division:

$$=2\int\left(-\frac{1}{2(s+1)}+\frac{1}{2(s-1)}+1\right)ds$$

Integrate the sum term by term and factor out constants:

$$=2\int 1\,ds+\int\frac{1}{s-1}\,ds-\int\frac{1}{s+1}\,ds$$

For the integrand $\frac{1}{s+1}$, substitute $p=s+1$ and $dp=ds$:

$$=-\int\frac{1}{p}\,dp+2\int 1\,ds+\int\frac{1}{s-1}\,ds$$

The integral of $\frac{1}{p}$ is $\log(p)$:

$$=-\log(p)+2\int 1\,ds+\int\frac{1}{s-1}\,ds$$

For the integrand $\frac{1}{s-1}$, substitute $w=s-1$ and $dw=ds$:

$$=-\log(p)+2\int 1\,ds+\int\frac{1}{w}\,dw$$

The integral of $\frac{1}{w}$ is $\log(w)$:

$$=-\log(p)+2\int 1\,ds+\log(w)$$

The integral of 1 is s:

$$=-\log(p)+2s+\log(w)+\text{constant}$$

Substitute back for $w=s-1$:

$$=-\log(p)+2s+\log(s-1)+\text{constant}$$

Substitute back for $p=s+1$:

$$=2s+\log(s-1)-\log(s+1)+\text{constant}$$

Substitute back for $s=\sqrt{u+1}$:

$$=2\sqrt{u+1}+\log\left(\sqrt{u+1}-1\right)-\log\left(\sqrt{u+1}+1\right)+\text{constant}$$

Substitute back for $u=\log(x)$:

$$=\log\left(\sqrt{\log(x)+1}-1\right)-\log\left(\sqrt{\log(x)+1}+1\right)+2\sqrt{\log(x)+1}+\text{constant}$$

An alternative form of the integral is:

$$=2\left(\sqrt{\log(x)+1}+\tanh^{-1}\left(\sqrt{\log(x)+1}\right)\right)+\text{constant}$$

Which is equivalent for restricted x values to:

$$=\log\left(-2\left(\sqrt{\log(x)+1}-1\right)\right)-\log\left(2\left(\sqrt{\log(x)+1}+1\right)\right)+2\sqrt{\log(x)+1}+\text{constant}$$

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6) $\int (e^{3x})^4 e^x \, dx =$

$\int e^{12x} e^x \, dx =$

$\int e^{13x} \, dx =$

$$\frac{e^{13x}}{13}$$

7

$$\int \sin\sqrt{x}\,dx$$

{substitute
$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}$ }

$$= 2\int u\sin(u)\,du$$

{substitute
$f = u$ $dg = \sin(u)\,du$
$df = du$ $g = -\cos(u)$ }

$$= 2\int \cos u\,du - 2u\cos(u)$$

$$= 2\sin(u) - 2u\cos(u) + c$$

{substitute $\sqrt{x}$ for u}

$$= 2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + c$$

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Integrate: $\int \frac{x^3}{x^4-1} dx$

Use the simple substitution of u =x^4 - 1. We find that du=$4x^3 dx$ and substitute in the appropriate values to get:

$$\int \frac{1}{4} \frac{du}{u} du$$

Using the fact that $\frac{dln(x)}{dx} = \frac{1}{x}$ we can integrate this equation to get:

$$\frac{ln(u)}{4} + C$$

By substituting x^4 - 1 for u we get the final answer:

$$\frac{ln(x^4 - 1)}{4} + C$$

#9 $\int \cos x \tan x\,dx = \int \cos x \frac{\sin x}{\cos x}\,dx$

$= \int \sin x\,dx = -\cos x + C.$

10.)

$$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx = \tan^{-1}(\sin(x)) + \text{constant}$$

Possible intermediate steps:

$$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx$$

For the integrand $\frac{\cos(x)}{\sin^2(x)+1}$, substitute $u = \sin(x)$ and $du = \cos(x)dx$:

$$= \int \frac{1}{u^2+1}\,du$$

The integral of $\frac{1}{u^2+1}$ is $\tan^{-1}(u)$:

$$= \tan^{-1}(u) + \text{constant}$$

Substitute back for $u = \sin(x)$:

$$= \tan^{-1}(\sin(x)) + \text{constant}$$

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11.

Possible intermediate steps:

$$\int x \sin^2(x)\, dx$$

For the integrand $x\sin^2(x)$, use the trigonometric identity $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$:

$$= \frac{1}{2}\int x(1-\cos(2x))\, dx$$

Expanding the integrand $x(1-\cos(2x))$ gives $x - x\cos(2x)$:

$$= \frac{1}{2}\int (x - x\cos(2x))\, dx$$

Integrate the sum term by term and factor out constants:

$$= \frac{1}{2}\int x\, dx - \frac{1}{2}\int x\cos(2x)\, dx$$

For the integrand $x\cos(2x)$, integrate by parts, $\int f\, dg = fg - \int g\, df$, where
$f = x, \quad dg = \cos(2x)dx,$
$df = dx, \quad g = \frac{1}{2}\sin(2x)$:

$$= -\frac{1}{4}x\sin(2x) + \frac{1}{2}\int x\, dx + \frac{1}{4}\int \sin(2x)\, dx$$

For the integrand $\sin(2x)$, substitute $u = 2x$ and $du = 2dx$:

$$= \frac{1}{8}\int \sin(u)\, du - \frac{1}{4}x\sin(2x) + \frac{1}{2}\int x\, dx$$

The integral of $\sin(u)$ is $-\cos(u)$:

$$= -\frac{\cos(u)}{8} - \frac{1}{4}x\sin(2x) + \frac{1}{2}\int x\, dx$$

The integral of x is $\frac{x^2}{2}$:

$$= -\frac{\cos(u)}{8} + \frac{x^2}{4} - \frac{1}{4}x\sin(2x) + \text{constant}$$

Substitute back for $u = 2x$:

$$= \frac{x^2}{4} - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x) + \text{constant}$$

(12) $\int \frac{\ln x + \sqrt{x}}{x}\,dx$

$\int \frac{\ln x}{x}\,dx + \int \frac{\sqrt{x}}{x}\,dx$ • separate terms

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\int u\,du + \int x^{-1/2}\,dx$ • use a u-substitution

$\frac{1}{2}u^2 + 2x^{1/2} + C$ • integrate

$\frac{1}{2}(\ln x)^2 + 2\sqrt{x} + C$ • put in terms of x

Problem 13

$$\int \frac{e^{2x}}{1+e^x} dx$$

Start by making the substitution u = e^x and du = e^x dx

$$\int \frac{u^2}{u(1+u)} du = \int \frac{u}{1+u} du$$

Modify the integral via long division

$$\int (1 - \frac{1}{1+u}) du$$

Split the main integral into two smaller ones

$$\int 1\, du - \int \frac{1}{1+u} du$$

Substitute w = 1 + u and dw = du

$$\int 1\, du - \int \frac{1}{w} dw$$

The integral of 1 is u, and the integral of $\frac{1}{w}$ is ln(w)

$$u - \ln(w) + C$$

Substitute 1 + u back in for w, and e^x back in for u

$$e^x - \ln(e^x + 1) + C$$

14

Possible intermediate steps:

$$\int \frac{\log(1+x)}{x^2}\, dx$$

For the integrand $\frac{\log(x+1)}{x^2}$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where

$f = \log(x+1), \quad dg = \frac{1}{x^2}\, dx,$

$df = \frac{1}{x+1}\, dx, \quad g = -\frac{1}{x}$:

$$= \int \frac{1}{x^2+x}\, dx - \frac{\log(x+1)}{x}$$

For the integrand $\frac{1}{x^2+x}$, complete the square:

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}\, dx - \frac{\log(x+1)}{x}$$

For the integrand $\frac{1}{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}$, substitute $u = x + \frac{1}{2}$ and $du = dx$:

$$= \int \frac{1}{u^2 - \frac{1}{4}}\, du - \frac{\log(x+1)}{x}$$

The integral of $\frac{1}{u^2 - \frac{1}{4}}$ is $-2\tanh^{-1}(2u)$:

$$= -2\tanh^{-1}(2u) - \frac{\log(x+1)}{x} + \text{constant}$$

Substitute back for $u = x + \frac{1}{2}$:

$$= -\frac{\log(x+1) + 2x\tanh^{-1}(2x+1)}{x} + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \log(x) - \frac{\log(x+1)}{x} - \log(x+1) + \text{constant}$$

15) $\int \frac{x^2}{\sqrt[3]{x-1}}\, dx$

x^2		$(x-1)^{-\frac{1}{3}}$
$2x$	$+$	$\frac{3(x-1)^{\frac{2}{3}}}{2}$
2	$-$	$\frac{9(x-1)^{\frac{5}{3}}}{10}$
0	$+$	$\frac{27(x-1)^{\frac{8}{3}}}{80}$

$$= \frac{3x^2(x-1)^{\frac{2}{3}}}{2} - \frac{9(x-1)^{\frac{5}{3}}}{10} + \frac{27(x-1)^{\frac{8}{3}}}{40} + C$$

16

$$\int (\sin x)(\cos(\cos x))\,dx$$

{ substitute: $u = \cos(x)$
$du = -\sin(x)$ }

$$= \int -\cos(u)\,du$$

$$= -\int \cos(u)\,du$$

$$= -\sin u + c$$

{ substitute cos x back in for u }

$$\boxed{= -\sin(\cos(x)) + c}$$

Wolfram Alpha

Page 368, Problem Number Seventeen.

Integrate: $\int \frac{\tan^{-1}(\sqrt{x})}{\sqrt{x}}\,dx$

We begin by making our u substitution. In this case it is easiest to say that u = $\sqrt{x}$.After a few calculations you will find that $du = \frac{1}{2\sqrt{x}}dx$. You thus have to multiply your integral by $2\sqrt{x}$ to eliminate the fraction created when you substitute du for dx. You should get:

$$2\int \tan^{-1} u\, du$$

Using the fact that the integral of $\tan^{-1} u = u\tan^{-1} u - \frac{1}{2}\ln(1 + u^2) + C$ you should be able to easily integrate this to get:

$$2(u\tan^{-1} u - \frac{1}{2}\ln(1 + u^2) + C)$$

By multiplying the two into the equation and replacing u with $\sqrt{x}$ you should get the final answer of:

$$\mathbf{2\sqrt{x}\tan^{-1}\sqrt{x} - \ln(1 + x) + C}$$

#18 $\int \sec^4 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx$

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

Let $u = \tan x$

$du = \sec^2 x\,dx$

$'' = \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C$

$= \tan x + \frac{1}{3}\tan^3 x + C$

19.)

$$\int \frac{3x+5}{x-2}\,dx = 3(x-2) + 11\log(x-2) + \text{constant}$$

Possible intermediate steps:

$$\int \frac{5+3x}{-2+x}\,dx$$

For the integrand $\frac{3x+5}{x-2}$, do long division:

$$= \int \left(\frac{11}{x-2} + 3\right) dx$$

Integrate the sum term by term and factor out constants:

$$= \int 3\,dx + 11 \int \frac{1}{x-2}\,dx$$

For the integrand $\frac{1}{x-2}$, substitute $u = x - 2$ and $du = dx$:

$$= 11 \int \frac{1}{u}\,du + \int 3\,dx$$

The integral of $\frac{1}{u}$ is $\log(u)$:

$$= 11\log(u) + \int 3\,dx$$

The integral of 3 is $3x$:

$$= 11\log(u) + 3x + \text{constant}$$

Substitute back for $u = x - 2$:

$$= 3x + 11\log(x-2) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= 3(x-2) + 11\log(x-2) + \text{constant}$$

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Possible intermediate steps:

#20

$\int (1+\sqrt{x})^8\,dx$

For the integrand $(\sqrt{x}+1)^8$, substitute $u=\sqrt{x}$ and $du=\frac{1}{2\sqrt{x}}\,dx$:

$=2\int u(u+1)^8\,du$

For the integrand $u(u+1)^8$, do long division:

$=2\int (u^9+8u^8+28u^7+56u^6+70u^5+56u^4+28u^3+8u^2+u)\,du$

Integrate the sum term by term and factor out constants:

$=2\int u^9\,du+16\int u^8\,du+56\int u^7\,du+112\int u^6\,du+$
$140\int u^5\,du+112\int u^4\,du+56\int u^3\,du+16\int u^2\,du+2\int u\,du$

The integral of u^2 is $\frac{u^3}{3}$:

$=2\int u^9\,du+16\int u^8\,du+56\int u^7\,du+112\int u^6\,du+$
$140\int u^5\,du+112\int u^4\,du+\frac{16u^3}{3}+56\int u^3\,du+2\int u\,du$

The integral of u^3 is $\frac{u^4}{4}$:

$=2\int u^9\,du+16\int u^8\,du+56\int u^7\,du+112\int u^6\,du+$
$140\int u^5\,du+14u^4+112\int u^4\,du+\frac{16u^3}{3}+2\int u\,du$

The integral of u^4 is $\frac{u^5}{5}$:

$=2\int u^9\,du+16\int u^8\,du+56\int u^7\,du+$
$112\int u^6\,du+\frac{112u^5}{5}+140\int u^5\,du+14u^4+\frac{16u^3}{3}+2\int u\,du$

The integral of u^5 is $\frac{u^6}{6}$:

$=2\int u^9\,du+16\int u^8\,du+56\int u^7\,du+$
$\frac{70u^6}{3}+112\int u^6\,du+\frac{112u^5}{5}+14u^4+\frac{16u^3}{3}+2\int u\,du$

The integral of u^6 is $\frac{u^7}{7}$:

$=2\int u^9\,du+16\int u^8\,du+16u^7+$
$56\int u^7\,du+\frac{70u^6}{3}+\frac{112u^5}{5}+14u^4+\frac{16u^3}{3}+2\int u\,du$

The integral of u^7 is $\frac{u^8}{8}$:

$=2\int u^9\,du+7u^8+16\int u^8\,du+$
$16u^7+\frac{70u^6}{3}+\frac{112u^5}{5}+14u^4+\frac{16u^3}{3}+2\int u\,du$

The integral of u^8 is $\frac{u^9}{9}$:

$=\frac{16u^9}{9}+2\int u^9\,du+7u^8+16u^7+\frac{70u^6}{3}+\frac{112u^5}{5}+14u^4+\frac{16u^3}{3}+2\int u\,du$

The integral of u^9 is $\frac{u^{10}}{10}$:

$=\frac{u^{10}}{5}+\frac{16u^9}{9}+7u^8+16u^7+\frac{70u^6}{3}+\frac{112u^5}{5}+14u^4+\frac{16u^3}{3}+2\int u\,du$

The integral of u is $\frac{u^2}{2}$:

$=\frac{u^{10}}{5}+\frac{16u^9}{9}+7u^8+16u^7+\frac{70u^6}{3}+\frac{112u^5}{5}+14u^4+\frac{16u^3}{3}+u^2+\text{constant}$

Substitute back for $u=\sqrt{x}$:

$=\frac{16x^{9/2}}{9}+16x^{7/2}+\frac{112x^{5/2}}{5}+$
$\frac{16x^{3/2}}{3}+\frac{x^5}{5}+7x^4+\frac{70x^3}{3}+14x^2+x+\text{constant}$

29.

(21) $\int \frac{\sqrt{4-x^2}}{x}\,dx$

$x = 2\cos u \qquad dx = 2\sin u\,du$

$\int \frac{\sqrt{4-4\cos^2 u}}{2\cos u} 2\sin u\,du$ • use a u-substitution

$2\int \frac{\sin^2 u}{\cos u}\,du$ • simplify terms

$2\int \frac{1-\cos^2 u}{\cos u}\,du$ • change $(\sin^2 u)$ to $(1-\cos^2 u)$

$2\int \sec u\,du - 2\int \cos u\,du$ • separate terms

$2\int \frac{\sec^2 u + \sec u\tan u}{\sec u + \tan u}\,du - 2\int \cos u\,du$ • multiply $(\sec u)$ by $\left(\frac{\sec u + \tan u}{\sec u + \tan u}\right)$

$v = \sec u + \tan u \qquad dv = (\sec^2 u + \sec u\tan u)\,du$

$2\int \frac{1}{v}\,dv - 2\int \cos u\,du$ • use a v-substitution

$2\ln(v) - 2\sin u + C$ • integrate

$2\ln(\sec u + \tan u) - 2\sin u + C$ • put in terms of u

$2\ln\left[\sec\left(\cos^{-1}\frac{x}{2}\right) + \tan\left(\cos^{-1}\frac{x}{2}\right)\right] - 2\sin\left(\cos^{-1}\frac{x}{2}\right) + C$ • put in terms of x

$2\ln\left(\frac{2}{x} + \frac{2}{x}\sqrt{1-\frac{x^2}{4}}\right) - 2\sqrt{1-\frac{x^2}{4}} + C$ • simplify terms

$2\ln\left(\frac{2+\sqrt{4-x^2}}{x}\right) - \sqrt{4-x^2} + C$ • simplify terms

Problem 22

$$\int \frac{1}{e^{2x}+5e^{x}}\,dx$$

Begin by making the substitution u = e^x and du = e^x dx

$$\int \frac{1}{u^2(u+5)}\,du$$

Transform the main integral using partial fractions

$$\int \left(\frac{1}{5u^2}+\frac{1}{25(u+5)}-\frac{1}{25u}\right)du$$

Split the main integral up into three smaller integrals

$$\frac{1}{5}\int \frac{1}{u^2}\,du+\frac{1}{25}\int \frac{1}{u+5}\,du-\frac{1}{25}\int \frac{1}{u}\,du$$

Make the substitution w = u + 5 and dw = du

$$\frac{1}{5}\int \frac{1}{u^2}\,du+\frac{1}{25}\int \frac{1}{w}\,dw-\frac{1}{25}\int \frac{1}{u}\,du$$

The integral of $\frac{1}{u^2}$ is $-\frac{1}{u}$, the integral of $\frac{1}{w}$ is ln(w), and the integral of $\frac{1}{u}$ is ln(u)

$$-\frac{1}{5u}+\frac{\ln(w)}{25}-\frac{\ln(u)}{25}+\ C$$

Substitute u + 5 back in for w and e^x for u

$$-\frac{1}{5e^{x}}+\frac{\ln(e^{x}+5)}{25}-\frac{\ln(e^{x})}{25}+\ C$$

23

Possible intermediate steps:

$$\int e^{-x^3} x^5 \, dx$$

For the integrand $e^{-x^3} x^5$, substitute $u = x^3$ and $du = 3x^2 dx$:

$$= \frac{1}{3} \int e^{-u} u \, du$$

For the integrand $e^{-u} u$, integrate by parts, $\int f \, dg = f g - \int g \, df$, where
$f = u, \quad dg = e^{-u} du,$
$df = du, \quad g = -e^{-u}$:

$$= \frac{1}{3} \int e^{-u} \, du - \frac{e^{-u} u}{3}$$

The integral of e^{-u} is $-e^{-u}$:

$$= -\frac{1}{3} e^{-u} u - \frac{e^{-u}}{3} + \text{constant}$$

Substitute back for $u = x^3$:

$$= -\frac{1}{3} e^{-x^3} x^3 - \frac{e^{-x^3}}{3} + \text{constant}$$

Which is equal to:

$$= \frac{1}{3} e^{-x^3} \left(-x^3 - 1\right) + \text{constant}$$

24) $\int (e^x + 1)^2 \, dx =$

$$\int \left(e^{2x} + 2e^x + 1\right) =$$

$$\frac{e^{2x}}{2} + 2e^x + x + C$$

25

$$\int \frac{x}{(x+3)^2}\,dx$$

{use partial fractions}

$$= \int \left(\frac{1}{x+3} - \frac{3}{(x+3)^2}\right) dx$$

$$= \int \frac{1}{x+3}\,dx - 3\int \frac{1}{(x+3)^2}\,dx$$

{substitute $u = x+3$ and $du = dx$}

$$= \int \frac{1}{x+3}\,dx - 3\int \frac{1}{u^2}\,du$$

$$= \frac{3}{u} + \int \frac{1}{x+3}\,dx$$ {substitute $s = x+3$ and $ds = dx$}

$$= \int \frac{1}{s}\,ds + \frac{3}{u}$$

$$= \log(s) + \frac{3}{u} + c$$

{substitute $x=3$ for s}

$$= \frac{3}{u} + \log(x+3) + c$$

{substitute $u = x+3$}

$$= \frac{(x+3)\log(x+3) + 3}{x+3} + c$$

$$\boxed{= \frac{3}{x+3} + \log(x+3) + c}$$

Wolfram Alpha

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Integrate: $\int x\sqrt[3]{x+5}\,dx$

By using the substitution, u = x + 5 you will find that $du = dx$ and after substituting in your u value you should get:

$$\int (u-5)\sqrt[3]{u}\,du$$

Convert $\sqrt[3]{u}$ into $u^{\frac{1}{3}}$ and multiply the terms out to get:

$$\int (u^{\frac{4}{3}} - 5u^{\frac{1}{3}})du$$

At this point you integrate the equation and should get:

$$\frac{3u^{\frac{7}{3}}}{7} - \frac{15u^{\frac{4}{3}}}{4} + C$$

You then replace u with x+5 to get the final answer of:

$$\frac{3(x+5)^{\frac{7}{3}}}{7} - \frac{15(x+5)^{\frac{4}{3}}}{4} + C$$

#27 $\int \frac{x}{x^4+2x^2+10}\,dx = \int \frac{x}{(x^2+1)^2+9}\,dx$

Let $3u = x^2+1$

$3\,du = 2x\,dx$

$\ldots = \frac{3}{2}\int \frac{du}{9u^2+9} = \frac{1}{6}\int \frac{du}{u^2+1}$

$= \frac{1}{6}\arctan u + C$

$= \frac{1}{6}\arctan \frac{x^2+1}{3} + C$

28.)

$$\int x^2 \sqrt{x^3 - 4}\, dx = \frac{2}{9} (x^3 - 4)^{3/2} + \text{constant}$$

Possible intermediate steps:

$$\int x^2 \sqrt{-4 + x^3}\, dx$$

For the integrand $x^2 \sqrt{x^3 - 4}$, substitute $u = x^3 - 4$ and $du = 3x^2 dx$:

$$= \frac{1}{3} \int \sqrt{u}\, du$$

The integral of $\sqrt{u}$ is $\frac{2u^{3/2}}{3}$:

$$= \frac{2u^{3/2}}{9} + \text{constant}$$

Substitute back for $u = x^3 - 4$:

$$= \frac{2}{9} (x^3 - 4)^{3/2} + \text{constant}$$

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Possible intermediate steps:

$$\int \frac{x \log(x)}{\sqrt{-1+x^2}}\, dx$$

For the integrand $\frac{x \log(x)}{\sqrt{x^2-1}}$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where

$f = \log(x), \quad dg = \frac{x}{\sqrt{x-1}}\, dx,$

$df = \frac{1}{x}\, dx, \quad g = \sqrt{x^2-1}$:

$$= \sqrt{x^2-1}\, \log(x) - \int \frac{\sqrt{x^2-1}}{x}\, dx$$

For the integrand, $\frac{\sqrt{x^2-1}}{x}$ substitute $x = \sec(u)$ and $dx = du \sec(u) \tan(u)$

. Then $\sqrt{x^2-1} = \sqrt{\sec^2(u)-1} = \tan(u)$ and $u = \sec^{-1}(x)$:

$$= \sqrt{x^2-1}\, \log(x) - \int \tan^2(u)\, du$$

The integral of $\tan^2(u)$ is $\tan(u) - u$:

$$= u - \tan(u) + \sqrt{x^2-1}\, \log(x) + \text{constant}$$

Substitute back for $u = \sec^{-1}(x)$:

$$= -\sqrt{1-\frac{1}{x^2}}\, x + \sqrt{x^2-1}\, \log(x) + \sec^{-1}(x) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \sqrt{x^2-1}\, (\log(x)-1) - \tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) + \text{constant}$$

(30) $\int \frac{\sin 2x}{\sqrt{4-\cos^4 x}}\,dx$

$\int \frac{\sin 2x}{\sqrt{4-\frac{1}{4}(1+\cos 2x)^2}}\,dx$

• change $(\cos^2 x)$ to $\frac{1}{2}(1+\cos 2x)$

$u = 1+\cos 2x \qquad du = 2\sin 2x$

$-\int \frac{1}{2\sqrt{4-\frac{1}{4}u^2}}\,du$

• use a u-substitution

$-\int \frac{1}{\sqrt{16-u^2}}\,du$

• apply (2) to $\left(\sqrt{4-\frac{1}{4}u^2}\right)$

$u = 4\sin v \qquad du = 4\cos v\,dv$

$-\int \frac{1}{\sqrt{1-\sin^2 v}}\cos v\,dv$

• use a v-substitution

$-\int \frac{\cos v}{\cos v}\,dv$

• change $\left(\sqrt{1-\sin^2 v}\right)$ to $(\cos v)$

$-\int dv$

• simplify

$-v + C$

• integrate

$-\sin^{-1}\left(\frac{u}{4}\right) + C$

• put in terms of u

$-\sin^{-1}\left(\frac{1+\cos 2x}{4}\right) + C$

• put in terms of x

Problem 31

$$\int x^2 \sin(x^3)\, dx$$

Begin by substituting u = x^3 and du = $3x^2dx$

$$\frac{1}{3}\int \sin(u)\, du$$

The integral of sin(u) is – cos(u)

$$-\frac{\cos(u)}{3} + C$$

Substitute x^3 back in for u

$$-\frac{\cos(x^3)}{3} + C$$

32

Possible intermediate steps:

$$\int x \sec(x) \tan(x)\, dx$$

For the integrand $x \sec(x) \tan(x)$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where
$f = x, \quad dg = \sec(x) \tan(x) dx,$
$df = dx, \quad g = \sec(x)$:

$$= x \sec(x) - \int \sec(x)\, dx$$

The integral of $\sec(x)$ is $\log(\sec(x) + \tan(x))$:

$$= x \sec(x) - \log(\tan(x) + \sec(x)) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= x \sec(x) + \log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) - \log\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) + \text{constant}$$

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33) $\int \frac{x}{(x^2+1)(x^2+4)}$

$$x = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$x = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$x = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

$0 = A + C$ | $0 = B + D$ | $1 = 4A + C$ | $0 = 4B + D$

$A = -C$ | $B = -D$ | $1 = -4C + C$ | $0 = -4D + D$

$0 = A - \frac{1}{3}$ | $B = 0$ | $1 = -3C$ | $0 = -3D$

$A = \frac{1}{3}$ | | $C = -\frac{1}{3}$ | $D = 0$

$$= \frac{1}{3}\int \frac{x}{(x^2+1)} - \frac{1}{3}\int \frac{x}{(x^2+4)} \, dx$$

= [illegible]

$u = x^2+1$ $u = x^2+4$

$du = 2x\,dx$ $du = 2x\,dx$

$$= \frac{1}{6}\int \frac{du}{u} - \frac{1}{6}\int \frac{du}{u} + C$$

$$= \frac{1}{6}\ln(x^2+1) - \frac{1}{6}\ln(x^2+4) + C$$

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$$\int \frac{dx}{x\sqrt{2x-16}}$$

$\{$substitute: $u = 2x-16$ and $du = 2dx\}$

$$= \int \frac{1}{\sqrt{u}\,(u+16)}\,du$$

$\left\{\text{substitute } s = \sqrt{u},\; ds = \frac{1}{2\sqrt{u}}\,du\right\}$

$$= 2\int \frac{1}{s^2+16}\,ds$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{s}{4}\right) + c$$

$\{$substitute $\sqrt{u} = s\}$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{u}}{4}\right) + c$$

$\{$substitute $u = 2x-16\}$

$$\boxed{= \frac{1}{2}\tan^{-1}\left(\frac{1}{4}\sqrt{2x-16}\right) + c}$$

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Page 368, Problem Number Thirty-Five

Integrate: $\int \frac{x^2}{(x-1)^3} dx$

This problem can be solved without a u-substitution by just partially decomposing the fraction inside of the integral. You should end up with:

$$\int \frac{x^2}{(x-1)^3} dx = \int \frac{A}{(x-1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} dx = \int \frac{(A)x^2+(B-2A)x+(A-B+C)}{(x-1)^3} dx$$

You then use the term on the far left $(x^2 + 0x + 0)$ to find out what A, B, and C are. You should get this series of equations and solutions:

$$\begin{cases} Ax^2 = x^2 : A = 1 \\ (B - 2A) = 0 : B = 2 \\ A - B + C = 0 : C = 1 \end{cases}$$

You then insert your A, B, and C values into the integral and integrate it. It is a bit simpler to do if you break the integral into three pieces as I have done below:

$$\int \frac{1}{(x-1)} dx + \int \frac{2}{(x-1)^2} dx + \int \frac{1}{(x-1)^3} dx$$

By integrating these equations you should end up with the final answer of:

$$\mathbf{\ln(x+1) - \frac{2}{x+1} - \frac{1}{2(x-1)^2} + c}$$

#36 $\int x^3 \ln x \, dx$

$u = \ln x \qquad du = \frac{1}{x} dx$

$dv = x^3 dx \qquad v = \frac{1}{4} x^4$

by parts ...

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx$

$= \frac{1}{4} x^4 \left(\ln x - \frac{1}{4} \right) + C$

37.)

$$\int \tan^3(x)\sec^4(x)\,dx = \frac{\sec^6(x)}{6} - \frac{\sec^4(x)}{4} + \text{constant}$$

Possible intermediate steps:

$$\int \sec^4(x)\tan^3(x)\,dx$$

For the integrand $\sec^4(x)\tan^3(x)$, use the trigonometric identity $\sec^2(x) = \tan^2(x) + 1$:

$$= \int \tan^3(x)\left(\tan^2(x) + 1\right)\sec^2(x)\,dx$$

For the integrand $\sec^2(x)\tan^3(x)\left(\tan^2(x) + 1\right)$
, substitute $u = \tan(x)$ and $du = \sec^2(x)dx$:

$$= \int u^3\left(u^2 + 1\right)du$$

For the integrand $u^3\left(u^2 + 1\right)$, do long division:

$$= \int\left(u^5 + u^3\right)du$$

Integrate the sum term by term:

$$= \int u^5\,du + \int u^3\,du$$

The integral of u^3 is $\frac{u^4}{4}$:

$$= \int u^5\,du + \frac{u^4}{4}$$

The integral of u^5 is $\frac{u^6}{6}$:

$$= \frac{u^6}{6} + \frac{u^4}{4} + \text{constant}$$

Substitute back for $u = \tan(x)$:

$$= \frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + \text{constant}$$

Factor the answer a different way:

$$= \frac{1}{12}\tan^4(x)\left(2\tan^2(x) + 3\right) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \frac{\sec^6(x)}{6} - \frac{\sec^4(x)}{4} + \text{constant}$$

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Possible intermediate steps:

$$\int (-e^{-x} + e^{x})^2 \, dx$$

Expanding the integrand $(-e^{-x} + e^{x})^2$ gives $-2 + e^{-2x} + e^{2x}$:

$$= \int \left(e^{-2x} + e^{2x} - 2\right) dx$$

Integrate the sum term by term:

$$= \int -2 \, dx + \int e^{-2x} \, dx + \int e^{2x} \, dx$$

The integral of e^{-2x} is $-\frac{1}{2} e^{-2x}$:

$$= -\frac{e^{-2x}}{2} + \int -2 \, dx + \int e^{2x} \, dx$$

The integral of e^{2x} is $\frac{e^{2x}}{2}$:

$$= -\frac{e^{-2x}}{2} + \frac{e^{2x}}{2} + \int -2 \, dx$$

The integral of -2 is $-2x$:

$$= -2x - \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} + \text{constant}$$

(39) $\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$

$x=\sin u \qquad dx=\cos u\,du$

$\int \frac{\sin u\cos u}{1-\sin^2 u+\sqrt{1-\sin^2 u}}\,du$ • use a u-substitution

$\int \frac{\sin u\cos u}{\cos^2 u+\cos u}\,du$ • change $(1-\sin^2 u)$ to $(\cos^2 u)$

$\int \frac{\sin u\cos u}{\cos u(\cos u+1)}\,du$ • factor out $(\cos u)$

$\int \frac{\sin u}{\cos u+1}\,du$ • simplify

$v=\cos u+1 \qquad dv=-\sin u\,du$

$-\int \frac{1}{v}\,dv$ • use a v-substitution

$-\ln v+C$ • integrate

$-\ln(\cos u+1)+C$ • put in terms of u

$-\ln[\cos(\sin^{-1}x)+1]+C$ • put in terms of x

$-(\ln(1+\sqrt{1-x^2})+C$ • simplify

Problem 40

$$\int x^3 e^{-2x} dx$$

Begin by integrating by parts, with f = x^3, df =$3x^2$ dx, dg = e^{-2x} dx, and g = $-\frac{1}{2}e^{-2x}$

$$-\frac{1}{2}e^{-2x}x^3 + \frac{3}{2}\int e^{-2x}x^2 dx$$

Integrate by parts again, with f = x^2, df =2x dx, dg = e^{-2x} dx, and g = $-\frac{1}{2}e^{-2x}$

$$-\frac{1}{2}e^{-2x}x^3 - \frac{3}{4}e^{-2x}x^2 + \frac{3}{2}\int e^{-2x}x dx$$

Integrate by parts a third time, with f = x, df = dx, dg = e^{-2x} dx, and g = $-\frac{1}{2}e^{-2x}$

$$-\frac{1}{2}e^{-2x}x^3 - \frac{3}{4}e^{-2x}x^2 - \frac{3}{4}e^{-2x}x + \frac{3}{4}\int e^{-2x} dx$$

The integral of $e^{-2x} = -\frac{1}{2}e^{-2x}$

$$-\frac{1}{2}e^{-2x}x^3 - \frac{3}{4}e^{-2x}x^2 - \frac{3}{4}e^{-2x}x - \frac{3}{8}e^{-2x} + C$$

41

Possible intermediate steps:

$\int \cos^4(x)\sin^2(x)\,dx$

Write $\sin^2(x)$ as $1-\cos^2(x)$:

$= \int \cos^4(x)\left(1-\cos^2(x)\right)dx$

Expanding the integrand $\cos^4(x)\left(1-\cos^2(x)\right)$ gives $\cos^4(x)-\cos^6(x)$:

$= \int \left(\cos^4(x)-\cos^6(x)\right)dx$

Integrate the sum term by term and factor out constants:

$= \int \cos^4(x)\,dx - \int \cos^6(x)\,dx$

Use the reduction formula, $\int \cos^m(x)\,dx = \frac{\cos^{m-1}(x)\sin(x)}{m} + \frac{m-1}{m}\int \cos^{-2+m}(x)\,dx$, where $m=6$:

$= \frac{1}{6}\int \cos^4(x)\,dx - \frac{1}{6}\sin(x)\cos^5(x)$

Use the reduction formula, $\int \cos^m(x)\,dx = \frac{\cos^{m-1}(x)\sin(x)}{m} + \frac{m-1}{m}\int \cos^{-2+m}(x)\,dx$, where $m=4$:

$= -\frac{1}{6}\sin(x)\cos^5(x) - \frac{5}{24}\sin(x)\cos^3(x) + \int \cos^4(x)\,dx - \frac{5}{8}\int \cos^2(x)\,dx$

Write $\cos^2(x)$ as $\frac{1}{2}\cos(2x)+\frac{1}{2}$:

$= -\frac{1}{6}\sin(x)\cos^5(x) - \frac{5}{24}\sin(x)\cos^3(x) +$
$\int \cos^4(x)\,dx - \frac{5}{8}\int\left(\frac{1}{2}\cos(2x)+\frac{1}{2}\right)dx$

Integrate the sum term by term and factor out constants:

$= -\frac{1}{6}\sin(x)\cos^5(x) - \frac{5}{24}\sin(x)\cos^3(x) -$
$\frac{5}{8}\int \frac{1}{2}\,dx + \int \cos^4(x)\,dx - \frac{5}{16}\int \cos(2x)\,dx$

For the integrand $\cos(2x)$, substitute $u=2x$ and $du=2dx$:

$= -\frac{5}{32}\int \cos(u)\,du - \frac{1}{6}\sin(x)\cos^5(x) -$
$\frac{5}{24}\sin(x)\cos^3(x) - \frac{5}{8}\int \frac{1}{2}\,dx + \int \cos^4(x)\,dx$

The integral of $\frac{1}{2}$ is $\frac{x}{2}$:

$= -\frac{5}{32}\int \cos(u)\,du - \frac{5x}{16} - \frac{1}{6}\sin(x)\cos^5(x) - \frac{5}{24}\sin(x)\cos^3(x) + \int \cos^4(x)\,dx$

The integral of $\cos(u)$ is $\sin(u)$:

$= -\frac{5\sin(u)}{32} - \frac{5x}{16} - \frac{1}{6}\sin(x)\cos^5(x) - \frac{5}{24}\sin(x)\cos^3(x) + \int \cos^4(x)\,dx$

Use the reduction formula, $\int \cos^m(x)\,dx = \frac{\cos^{m-1}(x)\sin(x)}{m} + \frac{m-1}{m}\int \cos^{-2+m}(x)\,dx$, where $m=4$:

$= -\frac{5\sin(u)}{32} - \frac{5x}{16} - \frac{1}{6}\sin(x)\cos^5(x) + \frac{1}{24}\sin(x)\cos^3(x) + \frac{3}{4}\int \cos^2(x)\,dx$

Write $\cos^2(x)$ as $\frac{1}{2}\cos(2x)+\frac{1}{2}$:

$= -\frac{5\sin(u)}{32} - \frac{5x}{16} - \frac{1}{6}\sin(x)\cos^5(x) +$
$\frac{1}{24}\sin(x)\cos^3(x) + \frac{3}{4}\int\left(\frac{1}{2}\cos(2x)+\frac{1}{2}\right)dx$

Integrate the sum term by term and factor out constants:

$= -\frac{5\sin(u)}{32} - \frac{5x}{16} - \frac{1}{6}\sin(x)\cos^5(x) +$
$\frac{1}{24}\sin(x)\cos^3(x) + \frac{3}{4}\int \frac{1}{2}\,dx + \frac{3}{8}\int \cos(2x)\,dx$

For the integrand $\cos(2x)$, substitute $s=2x$ and $ds=2dx$:

$= \frac{3}{16}\int \cos(s)\,ds - \frac{5\sin(u)}{32} - \frac{5x}{16} -$
$\frac{1}{6}\sin(x)\cos^5(x) + \frac{1}{24}\sin(x)\cos^3(x) + \frac{3}{4}\int \frac{1}{2}\,dx$

The integral of $\frac{1}{2}$ is $\frac{x}{2}$:

$= \frac{3}{16}\int \cos(s)\,ds - \frac{5\sin(u)}{32} + \frac{x}{16} - \frac{1}{6}\sin(x)\cos^5(x) + \frac{1}{24}\sin(x)\cos^3(x)$

The integral of $\cos(s)$ is $\sin(s)$:

$= \frac{3\sin(s)}{16} - \frac{5\sin(u)}{32} + \frac{x}{16} - \frac{1}{6}\sin(x)\cos^5(x) + \frac{1}{24}\sin(x)\cos^3(x) + \text{constant}$

Substitute back for $s=2x$:

$= -\frac{5\sin(u)}{32} + \frac{x}{16} - \frac{1}{6}\sin(x)\cos^5(x) +$
$\frac{1}{24}\sin(x)\cos^3(x) + \frac{3}{8}\sin(x)\cos(x) + \text{constant}$

Substitute back for $u=2x$:

$= \frac{x}{16} - \frac{5}{32}\sin(2x) - \frac{1}{6}\sin(x)\cos^5(x) +$
$\frac{1}{24}\sin(x)\cos^3(x) + \frac{3}{8}\sin(x)\cos(x) + \text{constant}$

Which is equal to:

$= \frac{x}{16} + \frac{1}{64}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{192}\sin(6x) + \text{constant}$

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42) $\int \frac{x}{\sqrt{1-4x^2}}\, dx$

$u = 1-4x^2$

$du = -8x$

$= -\frac{1}{8} \int \frac{du}{\sqrt{u}} = -\frac{1}{8} \int u^{-\frac{1}{2}}\, du$

$= -\frac{1}{8} \cdot 2\sqrt{u} + C$

$= -\frac{1}{4} \sqrt{1-4x^2} + C$

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$$\int \frac{\sqrt{x-1}}{x+3}$$

$$\left\{ \text{substitute } u = \sqrt{x-1} \quad du = \frac{1}{2\sqrt{x-1}}\,dx \right\}$$

$$= 2\int \frac{u^2}{u^2+4}\,du$$

$$= 2\int \left(1 - \frac{4}{u^2+4}\right) du$$

$$= 2\int 1\,du - 8\int \frac{1}{u^2+4}\,du$$

$$= 2\int 1\,du - 4\tan^{-1}\left(\frac{u}{2}\right)$$

$$= 2u - 4\tan^{-1}\left(\frac{u}{2}\right) + c$$

$$\left\{ \text{substitute } u = \sqrt{x-1} \right\}$$

$$\boxed{= 2\sqrt{x-1} - 4\tan^{-1}\left(\frac{\sqrt{x-1}}{2}\right) + c}$$

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Integrate: $\int \frac{x^3}{16+x^8} dx$

To put this problem into a form that you can integrate easily, factor out $\frac{1}{16}$ from the equation to get:

$$\frac{1}{16}\int \frac{x^3}{1+\frac{1}{16}x^8} dx$$

Note that this is in the form of $\frac{1}{1+x^2}$; which is the derivative of $\tan^{-1} x$, so you should integrate this and get:

$$\mathbf{\frac{1}{16}\tan^{-1}(\frac{1}{4}x^4)+C}$$

#45

$$\int \ln(x^2+3)\,dx$$

$$\left(\begin{array}{ll} u = \ln(x^2+3) & du = \frac{2x}{x^2+3} \\ dv = dx & v = x \end{array} \right)$$

by parts...

$$= x\ln(x^2+3) - 2\int \frac{x^2}{x^2+3}\,dx$$

Now work sub problem

$$\int \frac{x^2}{x^2+3}\,dx = \int \frac{x^2+3-3}{x^2+3}\,dx = \int \left(1 - \frac{3}{x^2+3}\right)dx$$

$$= x - 3\int \frac{1}{x^2+3}\,dx$$

$$\left(3p^2 = x^2,\ \sqrt{3}p = x,\ \sqrt{3}\,dp = dx\right)$$

$$\ldots = x - 3\int \frac{1}{3p^2+3}\sqrt{3}\,dp = x - \sqrt{3}\int \frac{1}{p^2+1}\,dp$$

$$= x - \sqrt{3}\arctan\frac{x}{\sqrt{3}} + C$$

Therefore

$$\int \ln(x^2+3)\,dx = x\ln(x^2+3) - 2x + 2\sqrt{3}\arctan\frac{x}{\sqrt{3}} + C$$

46.)

$$\int \frac{e^{\tan^{-1}(x)}}{1+x^2}\,dx = e^{\tan^{-1}(x)} + \text{constant}$$

Possible intermediate steps:

$$\int \frac{e^{\tan^{-1}(x)}}{1+x^2}\,dx$$

For the integrand $\frac{e^{\tan^{-1}(x)}}{x^2+1}$, substitute $u = \tan^{-1}(x)$ and $du = \frac{1}{x^2+1}\,dx$:

$$= \int e^u\,du$$

The integral of e^u is e^u:

$$= e^u + \text{constant}$$

Substitute back for $u = \tan^{-1}(x)$:

$$= e^{\tan^{-1}(x)} + \text{constant}$$

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47.

Possible intermediate steps:

$$\int e^5\, x \cos(3\,x)\, dx$$

Factor out constants:

$$= e^5 \int x \cos(3\,x)\, dx$$

For the integrand $x \cos(3\,x)$, integrate by parts, $\int f\, dg = f\, g - \int g\, df$, where
$f = x, \quad dg = \cos(3\,x) dx,$
$df = dx, \quad g = \frac{1}{3} \sin(3\,x)$:

$$= \frac{1}{3}\, e^5\, x \sin(3\,x) - \frac{e^5}{3} \int \sin(3\,x)\, dx$$

For the integrand $\sin(3\,x)$, substitute $u = 3\,x$ and $du = 3dx$:

$$= \frac{1}{3}\, e^5\, x \sin(3\,x) - \frac{e^5}{9} \int \sin(u)\, du$$

The integral of $\sin(u)$ is $-\cos(u)$:

$$= \frac{1}{9}\, e^5 \cos(u) + \frac{1}{3}\, e^5\, x \sin(3\,x) + \text{constant}$$

Substitute back for $u = 3\,x$:

$$= \frac{1}{3}\, e^5\, x \sin(3\,x) + \frac{1}{9}\, e^5 \cos(3\,x) + \text{constant}$$

Which is equal to:

$$= e^5 \left(\frac{1}{3}\, x \sin(3\,x) + \frac{1}{9} \cos(3\,x)\right) + \text{constant}$$

(48) $$\int \frac{x+1}{x^2-2x+2}\,dx$$

$$\int \frac{x-1}{x^2-2x+2}\,dx + \int \frac{2}{x^2-2x+2}\,dx$$

• separate terms

$$u = x^2-2x+2 \qquad du = (2x-2)\,dx$$

$$\frac{1}{2}\int \frac{1}{u}\,du + 2\int \frac{1}{x^2-2x+2}\,dx$$

• use a u-substitution

$$\frac{1}{2}\int \frac{1}{u}\,du + 2\int \frac{1}{1+(1-x)^2}\,dx$$

• simplify

$$v = 1-x \qquad dv = -dx$$

$$\frac{1}{2}\int \frac{1}{u}\,du - 2\int \frac{1}{1+v^2}\,dv$$

• use a v-substitution

$$\frac{1}{2}\ln u - 2\tan^{-1} v + c$$

• integrate

$$\frac{1}{2}\ln(x^2-2x+2) - 2\tan^{-1}(1-x) + c$$

• put in terms of x

Problem 49

$$\int \frac{x}{x^4 - 2x^2 - 3} dx$$

Begin by making the substitution u = x^2 and du =2xdx

$$\frac{1}{2}\int \frac{1}{u^2 - 2u - 3} du$$

Next complete the square to further simplify the integral

$$\frac{1}{2}\int \frac{1}{(u-1)^2 - 4} du$$

Make the substitution w = u—1 and dw = du

$$\frac{1}{2}\int \frac{1}{w^2 - 4} dw$$

The integral of $\frac{1}{w^2-4}$ is $\frac{\operatorname{arctanh}\left(\frac{w}{2}\right)}{2}$

$$\frac{1}{2} * \frac{\operatorname{arctanh}\left(\frac{w}{2}\right)}{2} + C$$

Substitute u – 1 back in for w, and x^2 for u

$$\frac{\operatorname{arctanh}\left(\frac{x^2 - 1}{2}\right)}{4} + C$$

50

Possible intermediate steps:

$$\int e^{e^x+x}\,dx$$

For the integrand e^{x+e^x}, substitute $u = e^x$ and $du = e^x\,dx$:

$$= \int e^u\,du$$

The integral of e^u is e^u:

$$= e^u + \text{constant}$$

Substitute back for $u = e^x$:

$$= e^{e^x} + \text{constant}$$

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51) $\int \frac{x^4+1}{x^5+5x+3}$

$u = x^5 + 5x + 3$

$du = 5x^4 + 5$

$= 5(x^4+1)$

$= \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln(u) + C$

$= \frac{1}{5}\ln(x^5+5x+3) + C$

52

$$\int x \sin^{-1}(x^2)\,dx$$

$\left\{ \text{substitute } u = x^2,\ du = 2x\,dx \right\}$

$$= \frac{1}{2}\int \sin^{-1}(u)\,du$$

$\left\{ \text{substitute: } f = \sin^{-1}(u),\ df = \frac{1}{\sqrt{1-u^2}}\,du,\ dg = du,\ g = u \right\}$

$$= \frac{1}{2}u\sin^{-1}(u) - \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}}\,du$$

$\left\{ \text{substitute } s = 1-u^2,\ ds = -2u\,du \right\}$

$$= \frac{1}{4}\int \frac{1}{\sqrt{s}}\,ds + \frac{1}{2}u\sin^{-1}(u)$$

$$= \frac{\sqrt{s}}{2} + \frac{1}{2}u\sin^{-1}(u) + C$$

$\left\{ \text{substitute } s = 1-u^2 \right\}$

$$= \frac{\sqrt{1-u^2}}{2} + \frac{1}{2}u\sin^{-1}(u) + C$$

$\left\{ \text{substitute } u = x^2 \right\}$

$$= \frac{\sqrt{1-x^4}}{2} + \frac{1}{2}x^2\sin^{-1}(x^2) + C$$

$$\boxed{= \frac{1}{2}\left(\sqrt{1-x^4} + x^2\sin^{-1}(x^2)\right) + C}$$

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Page 368, Problem Number Fifty-Three

Integrate: $\int \frac{dx}{x+7+5\sqrt{x+1}}$

Factor the denominator. Substitute u = $\sqrt{x+1}$ into the resulting equation. $du = \frac{1}{2\sqrt{x+1}} dx$, so you should get:

$$2 \int \frac{udu}{(u+3)(u+2)}$$

You should decompose this fraction to get the following integral, and answers to your decomposition equations:

$$2 \int \frac{A}{(u+3)} + \frac{B}{(u+2)} du$$

$$\begin{cases} A + B = 1 \\ 2A + 3B = 0 \end{cases}$$

Solve to get:

$$\begin{cases} A = 1 - B \\ 2(1 - B) + 3B = 0 : \boldsymbol{B = -2} \\ \boldsymbol{A = 3} \end{cases}$$

Insert your A and B values in the appropriate place and use the fact that $\frac{dln(x)}{dx} = \frac{1}{x}$ to integrate this equation and get your final answer. I will show the portion from where you plug in A and B to where you get the final answer:

$$2 \int \frac{3}{(u+3)} + \frac{-2}{(u+2)} du = 2(3 \ln(u + 3) - 2 \ln(u + 2)) =$$

$$\mathbf{6 \ln\left(\sqrt{x+1} + 3\right) - 4 \ln\left(\sqrt{x+1} + 2\right) + C}$$

#54

$$\int \frac{1}{x^3+x^2+x+1}\,dx = \int \frac{x-1}{x^4-1}\,dx$$

$$= \int \frac{x-1}{(x^2-1)(x^2+1)}\,dx = \int \frac{1}{(x+1)(x^2+1)}\,dx$$

Partial fractions.

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$1 = (x^2+1)A + (x+1)(Bx+C)$$

at $x=0$ $\quad 1 = A + C \quad C = 1 - A$

at $x=-1$ $\quad 2A = 1 \quad A = 1/2$

$C = 1/2$

differentiate

$$0 = 2xA + (2x+1)B + C$$

at $x=0$ $\quad 0 = B + C \quad B = -1/2$

Therefore

$$\ldots = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan x + C$$

55.)

$$\int \frac{\sin(x)}{1+3\cos^2(x)}\,dx = -\frac{\tan^{-1}\left(\sqrt{3}\,\cos(x)\right)}{\sqrt{3}} + \text{constant}$$

Possible intermediate steps:

$$\int \frac{\sin(x)}{1+3\cos^2(x)}\,dx$$

For the integrand $\frac{\sin(x)}{3\cos^2(x)+1}$, substitute $u = \cos(x)$ and $du = -\sin(x)dx$:

$$= -\int \frac{1}{3u^2+1}\,du$$

The integral of $\frac{1}{3u^2+1}$ is $\frac{\tan^{-1}\left(\sqrt{3}\,u\right)}{\sqrt{3}}$:

$$= -\frac{\tan^{-1}\left(\sqrt{3}\,u\right)}{\sqrt{3}} + \text{constant}$$

Substitute back for $u = \cos(x)$:

$$= -\frac{\tan^{-1}\left(\sqrt{3}\,\cos(x)\right)}{\sqrt{3}} + \text{constant}$$

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56.

Possible intermediate steps:

$$\int \frac{3+2x}{1+x^2}\, dx$$

Expanding the integrand $\frac{2x+3}{x^2+1}$ gives $\frac{2x}{x^2+1} + \frac{3}{x^2+1}$:

$$= \int \left(\frac{2x}{x^2+1} + \frac{3}{x^2+1}\right) dx$$

Integrate the sum term by term and factor out constants:

$$= 3\int \frac{1}{x^2+1}\, dx + 2\int \frac{x}{x^2+1}\, dx$$

For the integrand $\frac{x}{x^2+1}$, substitute $u = x^2+1$ and $du = 2x\,dx$:

$$= \int \frac{1}{u}\, du + 3\int \frac{1}{x^2+1}\, dx$$

The integral of $\frac{1}{x^2+1}$ is $\tan^{-1}(x)$:

$$= \int \frac{1}{u}\, du + 3\tan^{-1}(x)$$

The integral of $\frac{1}{u}$ is $\log(u)$:

$$= \log(u) + 3\tan^{-1}(x) + \text{constant}$$

Substitute back for $u = x^2+1$:

$$= \log\left(x^2+1\right) + 3\tan^{-1}(x) + \text{constant}$$

(57) $\int \frac{x^3}{(x+1)^8}\,dx$

$\int \left(\frac{1}{(x+1)^5} - \frac{3}{(x+1)^6} + \frac{3}{(x+1)^7} - \frac{1}{(x+1)^8} \right) dx$	• divide up term
$u = x+1 \qquad du = dx$	
$\int (u^{-5} - 3u^{-6} + 3u^{-7} - u^{-8})\,du$	• use a u-substitution
$-\frac{1}{4}u^{-4} + \frac{3}{5}u^{-5} - \frac{1}{2}u^{-6} + \frac{1}{7}u^{-7} + C$	• integrate
$-\frac{1}{4}(x+1)^{-4} + \frac{3}{5}(x+1)^{-5} - \frac{1}{2}(x+1)^{-6} + \frac{1}{7}(x+1)^{-7} + C$	• put in terms of x

Problem 58

$$\int \sin(\ln(x))dx$$

Begin by making the substitution u = ln(x) du = $\frac{1}{x}$dx, so e^u = x and e^u du = dx

$$\int \sin(u)e^u du$$

Integrate by parts, with f = sin(u), df =cos(u)du, dg = e^u du, and g = e^u

$$\sin(u)\, e^u - \int \cos(u)e^u du$$

Integrate by parts again, with f = cos(u), df =—sin(u)du, dg = e^u du, and g = e^u

$$\sin(u)\, e^u - (\cos(u)\, e^u - \int \sin(u)e^u du)$$

Because our original integral was $\int \sin(u)\, e^u du$, we cannot simplify by repeated integration by parts

$$\sin(u)\, e^u - (\cos(u)\, e^u - \int \sin(u)e^u du) = \int \sin(u)e^u du$$

Instead we will manipulate the equation by basic algebra to get the answer

$$\sin(u)\, e^u - \cos(u)\, e^u = 2\int \sin(u)e^u du$$

$$\frac{\sin(u)\, e^u - \cos(u)\, e^u}{2} = \int \sin(u)e^u du$$

Substitute ln(x) back in for u, using e^u = x

$$\frac{x\sin(\ln(x)) - x\cos(\ln(x))}{2} + C$$

59

Possible intermediate steps:

$$\int \tan^6(x)\,dx$$

Use the reduction formula, $\int \tan^m(x)\,dx = \frac{\tan^{m-1}(x)}{m-1} - \int \tan^{-2+m}(x)\,dx$, where $m = 6$:

$$= \frac{\tan^5(x)}{5} - \int \tan^4(x)\,dx$$

Use the reduction formula, $\int \tan^m(x)\,dx = \frac{\tan^{m-1}(x)}{m-1} - \int \tan^{-2+m}(x)\,dx$, where $m = 4$:

$$= \frac{\tan^5(x)}{5} - \frac{\tan^3(x)}{3} + \int \tan^2(x)\,dx$$

The integral of $\tan^2(x)$ is $\tan(x) - x$:

$$= -x + \frac{\tan^5(x)}{5} - \frac{\tan^3(x)}{3} + \tan(x) + \text{constant}$$

Which is equal to:

$$= -x + \frac{23\tan(x)}{15} + \frac{1}{5}\tan(x)\sec^4(x) - \frac{11}{15}\tan(x)\sec^2(x) + \text{constant}$$

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60) $\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx$

$u = \sin x - \cos x$

$du = \cos x + \sin x \, dx$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|\sin x - \cos x| + C$

61

$$\int \frac{x}{x^2+5x+6}\,dx$$

$$= \int \left(\frac{2x+5}{2(x^2+5x+6)} - \frac{5}{2(x^2+5x+6)} \right) dx$$

$$= \frac{1}{2} \int \frac{2x+5}{x^2+5x+6}\,dx - \frac{5}{2} \int \frac{1}{x^2+5x+6}\,dx$$

{substitute $u = x^2+5x+6$, $du = 2x+5\,dx$}

$$= \frac{1}{2} \int \frac{1}{u}\,du - \frac{5}{2} \int \frac{1}{x^2+5x+6}\,dx$$

$$= \frac{1}{2} \int \frac{1}{u}\,du - \frac{5}{2} \int \frac{1}{(x+\frac{5}{2})^2 - \frac{1}{4}}\,dx$$

{substitute $s = x+\frac{5}{2}$ and $ds = dx$}

$$= \frac{1}{2} \int \frac{1}{u}\,du - \frac{5}{2} \int \frac{1}{s^2 - \frac{1}{4}}\,ds$$

$$= 5\tanh^{-1}(2s) + \frac{\log(u)}{2} + c$$

{substitute $s = x+\frac{5}{2}$}

$$= \frac{\log(u)}{2} + 5\tanh^{-1}(2x+5) + c$$

{substitute $u = x^2+5x+5$}

$$= \frac{1}{2}\log(x^2+5x+6) + 5\tanh^{-1}(2x+5) + c$$

$$\boxed{= 3\log(-x-3) - 3\log(x+2) + c}$$

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Page 368, Problem Number Sixty-Two

Integrate: $\int \frac{dx}{\sqrt{1-4x^2}}$:

To solve this integral simply plug in $\frac{1}{2}\sin u$ for x. You should note that $dx=\frac{1}{2}\cos u\, du$. We now substitute in our u value and ensure equality when we switch $du\ and\ dx$ and get:

$$\frac{1}{2}\int \frac{\cos u\, du}{\sqrt{1-4(\frac{1}{2}\sin u)^2}}$$

When the $\frac{1}{2}$ in the denominator is squared it becomes $\frac{1}{4}$ to cancel out the four in the denominator. We then end up with:

$$\frac{1}{2}\int \frac{\cos u du}{\sqrt{1-\sin^2 u}}$$

By using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ you get:

$$\frac{1}{2}\int \frac{\cos u du}{\sqrt{\cos^2 u}}$$

$\sqrt{\cos^2 u} = \cos u$. Using this fact we notice that the $\cos u$ in the numerator cancels-out the one in the denominator. We now have:

$$\int \frac{1}{2} du$$

We integrate this equation to get:

$$\frac{1}{2}u + \mathrm{C}$$

Remember that we originally substituted $\frac{1}{2}\sin u$ for x. To put u in terms of x we simply use $\frac{1}{2}\sin u = x$ to solve for x and get $\sin^{-1} 2x$. We substitute this for u in our integral to get our final answer of:

$$\mathbf{\frac{1}{2}\sin^{-1}(2x) + C}$$

#63

$$\int \ln\sqrt{2x-1}\,dx = \int \ln(2x-1)^{1/2}\,dx$$

$$= \frac{1}{2}\int \ln(2x-1)\,dx$$

$$u = 2x-1$$
$$du = 2\,dx$$

$$= \frac{1}{4}\int \ln u\,du$$

$$= \frac{1}{4}(u\ln u - u) + C_1$$

$$= \frac{1}{4}(2x-1)\ln(2x-1) - \frac{1}{4}(2x-1) + C_1$$

$$= \frac{1}{4}(2x-1)\ln(2x-1) - \frac{1}{2}x + C$$

64.)

$$\int \frac{4}{x^2 + 4x + 20}\, dx = \tan^{-1}\left(\frac{x+2}{4}\right) + \text{constant}$$

Possible intermediate steps:

$$\int \frac{4}{20+4x+x^2}\, dx$$

Factor out constants:

$$= 4\int \frac{1}{x^2+4x+20}\, dx$$

For the integrand $\frac{1}{x^2+4x+20}$, complete the square:

$$= 4\int \frac{1}{(x+2)^2+16}\, dx$$

For the integrand $\frac{1}{(x+2)^2+16}$, substitute $u = x+2$ and $du = dx$:

$$= 4\int \frac{1}{u^2+16}\, du$$

The integral of $\frac{1}{u^2+16}$ is $\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)$:

$$= \tan^{-1}\left(\frac{u}{4}\right) + \text{constant}$$

Substitute back for $u = x+2$:

$$= \tan^{-1}\left(\frac{x+2}{4}\right) + \text{constant}$$

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#65

Possible intermediate steps:

$$\int \sqrt{\frac{1+x}{-1+x}}\, dx$$

For the integrand $\sqrt{\frac{x+1}{x-1}}$, substitute $u = \frac{x+1}{x-1}$ and $du = \frac{1}{x-1} - \frac{x+1}{(x-1)^2}\, dx$:

$$= -2 \int \frac{\sqrt{u}}{(1-u)^2}\, du$$

For the integrand $\frac{\sqrt{u}}{(1-u)^2}$, substitute $s = \sqrt{u}$ and $ds = \frac{1}{2\sqrt{u}}\, du$:

$$= -4 \int \frac{s^2}{(1-s^2)^2}\, ds$$

For the integrand, $\frac{s^2}{(1-s^2)^2}$ substitute $s = \sin(p)$ and $ds = \cos(p)\, dp$. Then $(1-s^2)^2 = (1-\sin^2(p))^2 = \cos^4(p)$ and $p = \sin^{-1}(s)$:

$$= -4 \int \tan^2(p) \sec(p)\, dp$$

For the integrand $\sec(p)\tan^2(p)$, write $\tan^2(p)$ as $\sec^2(p) - 1$:

$$= -4 \int \sec(p)\left(\sec^2(p) - 1\right) dp$$

Expanding the integrand $\sec(p)\left(\sec^2(p) - 1\right)$ gives $\sec^3(p) - \sec(p)$:

$$= -4 \int \left(\sec^3(p) - \sec(p)\right) dp$$

Integrate the sum term by term and factor out constants:

$$= 4 \int \sec(p)\, dp - 4 \int \sec^3(p)\, dp$$

Use the reduction formula, $\int \sec^m(p)\, dp = \frac{\sec^{m-1}(p)\sin(p)}{m-1} + \frac{m-2}{m-1} \int \sec^{-2+m}(p)\, dp$, where $m = 3$:

$$= 2 \int \sec(p)\, dp - 2\tan(p)\sec(p)$$

The integral of $\sec(p)$ is $\log(\sec(p) + \tan(p))$:

$$= -2\tan(p)\sec(p) + 4 \int \sec(p)\, dp - 2\log(\tan(p) + \sec(p))$$

The integral of $\sec(p)$ is $\log(\sec(p) + \tan(p))$:

$$= 2\log(\tan(p) + \sec(p)) - 2\tan(p)\sec(p) + \text{constant}$$

Substitute back for $p = \sin^{-1}(s)$:

$$= \frac{2\left((s^2-1)\log\left(\frac{\sqrt{s+1}}{\sqrt{1-s}}\right) + s\right)}{s^2-1} + \text{constant}$$

Substitute back for $s = \sqrt{u}$:

$$= \frac{2\left(\sqrt{u} + (u-1)\log\left(\frac{\sqrt{\sqrt{u}+1}}{\sqrt{1-\sqrt{u}}}\right)\right)}{u-1} + \text{constant}$$

Substitute back for $u = \frac{x+1}{x-1}$:

$$= \frac{\sqrt{-x-1}\,(x-1) + 2\sqrt{1-x}\,\log\left(\frac{\sqrt{\sqrt{\frac{x+1}{x-1}}+1}}{\sqrt{\frac{x+\sqrt{-x-1}\,\sqrt{1-x}-1}{x-1}}}\right)}{\sqrt{1-x}} + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \frac{\sqrt{\frac{x+1}{x-1}}\left(\sqrt{x+1}\,(x-1) + 2\sqrt{x-1}\,\sinh^{-1}\left(\frac{\sqrt{x-1}}{\sqrt{2}}\right)\right)}{\sqrt{x+1}} + \text{constant}$$

3.

(66) $\int \frac{x}{\sqrt{16-x^4}}\,dx$

$u = x^2 \qquad du = 2x\,dx$

$\frac{1}{2}\int \frac{1}{\sqrt{16^2-u^2}}\,du$ — • use a u-substitution

$u = 4\sin v \qquad du = 4\cos v\,dv$

$\frac{1}{2}\int \frac{1}{\sqrt{1-\sin^2 v}}\cos v\,dv$ — • use a v-substitution

$\frac{1}{2}\int \frac{\cos v}{\cos v}\,dv$ — • change $(\sqrt{1-\sin^2 v})$ to $(\cos v)$

$\frac{1}{2}\int dv$ — • simplify

$\frac{1}{2}v + C$ — • integrate

$\frac{1}{2}\sin^{-1}\left(\frac{u}{4}\right) + C$ — • put in terms of u

$\frac{1}{2}\sin^{-1}\left(\frac{x^2}{4}\right) + C$ — • put in terms of x

Problem 67

$$\int \sin^2(5x)\cos^2(5x)\,dx$$

Begin by making the substitution u = 5x and du = 5dx

$$\frac{1}{5}\int \sin^2(u)\cos^2(u)\,du$$

Rewrite $\cos^2(u)$ as $1-\sin^2(u)$ and multiply out the $\sin^2(u)$

$$\frac{1}{5}\int \sin^2(u) - \sin^4(u)\,du$$

Rewrite $\sin^2(u)$ as $\frac{1-\cos(2u)}{2}$ and split the main integral up into three smaller ones

$$\frac{1}{5}\int \frac{1-\cos(2u)}{2} - \sin^4(u)\,du = \frac{1}{5}\int \frac{1}{2}du - \frac{1}{10}\int \cos(2u)\,du - \frac{1}{5}\int \sin^4(u)du$$

For the second integral, make the substitution w = 2u so dw = 2du

$$\frac{1}{5}\int \frac{1}{2}du - \frac{1}{20}\int \cos(w)\,dw - \frac{1}{5}\int \sin^4(u)du$$

The reduction formula for $\int \sin^n(u)\,du = -\frac{\sin^{n-1}(u)\cos(u)}{n} + \frac{n-1}{n}\int \sin^{-2+n}(u)du$, with n = 4 in this case

$$\frac{1}{5}\int \frac{1}{2}du - \frac{1}{20}\int \cos(w)\,dw + \frac{1}{20}\sin^3(u)\cos(u)du - \frac{3}{20}\int \sin^2(u)du$$

Rewrite $\sin^2(u)$ as $\frac{1-\cos(2u)}{2}$ and split the integral up into two smaller ones

$$\frac{1}{20}\int \frac{1}{2}du - \frac{1}{20}\int \cos(w)\,dw + \frac{1}{20}\sin^3(u)\cos(u)du + \frac{3}{40}\int \cos(2u)du$$

In the third integral, substitute w = 2u and dw = 2du

$$\frac{1}{20}\int \frac{1}{2}du - \frac{1}{20}\int \cos(w)\,dw + \frac{1}{20}\sin^3(u)\cos(u)du + \frac{3}{80}\int \cos(w)dw$$

The integral of $\frac{1}{2}$ is $\frac{u}{2}$, and the integral of cos(w) is sin(w)

$$\frac{u}{40} - \frac{\sin(w)}{20} + \frac{3\sin(w)}{80} + \frac{\sin^3(u)\cos(u)}{20} + C$$

Substitute back 2u for w, 5x for u, and then simplify

$$\frac{5x}{40} - \frac{\sin(10x)}{20} + \frac{3\sin(10x)}{80} + \frac{\sin^3(5x)\cos(5x)}{20} + C = \frac{x}{8} - \frac{\sin(10x)}{80} + \frac{\sin^3(5x)\cos(5x)}{20} + C$$

68

Possible intermediate steps:

$$\int \frac{1}{-6+5x+x^2}\,dx$$

For the integrand $\frac{1}{x^2+5x-6}$, complete the square:

$$= \int \frac{1}{\left(x+\frac{5}{2}\right)^2 - \frac{49}{4}}\,dx$$

For the integrand $\frac{1}{\left(x+\frac{5}{2}\right)^2 - \frac{49}{4}}$, substitute $u = x + \frac{5}{2}$ and $du = dx$:

$$= \int \frac{1}{u^2 - \frac{49}{4}}\,du$$

The integral of $\frac{1}{u^2 - \frac{49}{4}}$ is $-\frac{2}{7}\tanh^{-1}\left(\frac{2u}{7}\right)$:

$$= -\frac{2}{7}\tanh^{-1}\left(\frac{2u}{7}\right) + \text{constant}$$

Substitute back for $u = x + \frac{5}{2}$:

$$= -\frac{2}{7}\tanh^{-1}\left(\frac{2x}{7} + \frac{5}{7}\right) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \frac{1}{7}\log(1-x) - \frac{1}{7}\log(x+6) + \text{constant}$$

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69) $\int \cot x \ln(\sin x)\, dx$

$u = \ln(\sin x)$

$du = \cot x\, dx$

$= \int u\, du = \frac{1}{2} u^2 + C$

$= \frac{1}{2}[\ln(\sin x)]^2 + C$

70

$$\int \frac{dx}{e^{3x}-e^{x}} \quad \left\{ \text{substitute: } \begin{matrix} u=e^{x} \\ du=e^{x}dx \end{matrix} \right\}$$

$$= \int \frac{1}{u^2(u^2-1)}\,du$$

$$= \int \left(\frac{1}{-u^2} - \frac{1}{2(u+1)} + \frac{1}{2(u-1)} \right) du$$

$$= -\int \frac{1}{u^2}\,du + \frac{1}{2}\int \frac{1}{u-1}\,du - \frac{1}{2}\int \frac{1}{u+1}\,du$$

{substitute $s=u-1$ and $ds=du$}

$$= \frac{1}{2}\int \frac{1}{s}\,ds - \int \frac{1}{u^2}\,du - \frac{1}{2}\int \frac{1}{u+1}\,du$$

{substitute $p=u+1$ and $dp=du$}

$$= -\frac{1}{2}\int \frac{1}{p}\,dp + \frac{1}{2}\int \frac{1}{s}\,ds - \int \frac{1}{u^2}\,du$$

$$= \frac{-\log(p)}{2} + \frac{1}{2}\int \frac{1}{s}\,ds - \int \frac{1}{u^2}\,du$$

$$= \frac{-\log(p)}{2} + \frac{\log(s)}{2} + \frac{1}{u} + C$$

{substitute: $p=u-1$ and $s=u-1$}

$$= \frac{1}{u} + \frac{1}{2}\log(u-1) - \frac{1}{2}\log(u+1) + C$$

substitute: $u=e^{x}$

$$= e^{-x} + \frac{1}{2}\log(e^{x}-1) - \frac{1}{2}\log(e^{x}+1) + C$$

$$\boxed{= e^{-x} + \tanh^{-1}(e^{x}) + C}$$

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Page 368, Problem Number Seventy-One

Integrate: $\int \csc^3(2x)\cot^3(2x)\,dx$

The best way to begin this problem is by setting u = $\csc 2x$. You should find that $du = -2\csc 2x \cot 2x\,dx$. Do not plug in du just yet. Rearrange your equation so that you get it in this form:

$$\int \csc^2(2x)\cot^2(2x)\cdot(\csc(2x)\cot(2x)\,dx)$$

Use the form of the Pythagorean Identity that relates cot and csc to put your equation in the form of:

$$\int \csc^2(2x)(\csc^2(2x)-1)\cdot(\csc(2x)\cot(2x)\,dx)$$

$$= \int [\csc^4(2x)-\csc^2(2x)]\cdot(\csc(2x)\cot(2x)\,dx)$$

You should now substitute u into the equation, and ensure equality by eliminating the term that is being multiplied by dx and by multiplying the integral by $-\frac{1}{2}$

$$-\frac{1}{2}\int (u^4-u^2)\cdot du$$

Integrate this equation to get:

$$-\frac{u^5}{10}+\frac{u^3}{6}+C$$

Put u in terms of x to get the final answer of:

$$\frac{\csc^3 2x}{6}-\frac{\csc^5 2x}{10}+C$$

#72 $\int \frac{e^x+1}{e^x-1}\,dx$

$$u = e^x$$

$$du = e^x dx = u\,dx$$

$$\ldots = \int \frac{u+1}{u-1}\,\frac{1}{u}\,du = \int \frac{u+1}{u(u-1)}\,du$$

Partial fractions...

$$\frac{u+1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$u+1 = (u-1)A + uB$$

at $u=0$ $\quad 1 = -1A \quad$ so $A=-1$

at $u=1$ $\quad 2 = B \quad$ so $B=2$

$$\ldots = \int \left(\frac{-1}{u} + \frac{2}{u-1}\right) du$$

$$= -\ln|u| + 2\ln|u-1| + C$$

$$= -\ln e^x + 2\ln|e^x-1| + C$$

$$= -x + \ln(e^x-1)^2 + C$$

73.)

$$\int \log\left(2\,x+x^2\right)d\,x = -2\,x + x\log(x\,(x+2)) + 2\log(x+2) + \text{constant}$$

Log is the natural logarithm

Possible intermediate steps:

$$\int \log\left(2\,x+x^2\right)d\,x$$

For the integrand $\log\left(x^2+2\,x\right)$, integrate by parts, $\int f\,d\,g = f\,g - \int g\,d\,f$, where

$f = \log\left(x^2+2\,x\right), \quad d\,g = d\,x,$

$d\,f = \frac{2\,(x+1)}{x\,(x+2)}\,d\,x, \quad g = x:$

$$= x\log\left(x^2+2\,x\right) - \int \frac{2\,(x+1)}{x+2}\,d\,x$$

Factor out constants:

$$= x\log\left(x^2+2\,x\right) - 2\int \frac{x+1}{x+2}\,d\,x$$

For the integrand $\frac{x+1}{x+2}$, do long division:

$$= x\log\left(x^2+2\,x\right) - 2\int \left(1-\frac{1}{x+2}\right)d\,x$$

Integrate the sum term by term and factor out constants:

$$= x\log\left(x^2+2\,x\right) - 2\int 1\,d\,x + 2\int \frac{1}{x+2}\,d\,x$$

For the integrand $\frac{1}{x+2}$, substitute $u = x+2$ and $d\,u = d\,x$:

$$= 2\int \frac{1}{u}\,d\,u + x\log\left(x^2+2\,x\right) - 2\int 1\,d\,x$$

The integral of $\frac{1}{u}$ is $\log(u)$:

$$= 2\log(u) + x\log\left(x^2+2\,x\right) - 2\int 1\,d\,x$$

The integral of 1 is x:

$$= 2\log(u) + x\log\left(x^2+2\,x\right) - 2\,x + \text{constant}$$

Substitute back for $u = x+2$:

$$= -2\,x + x\log(x\,(x+2)) + 2\log(x+2) + \text{constant}$$

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#74

Possible intermediate steps:

$$\int \csc(4x)\, dx$$

For the integrand $\csc(4x)$, substitute $u = 4x$ and $du = 4dx$:

$$= \frac{1}{4} \int \csc(u)\, du$$

The integral of $\csc(u)$ is $-\log(\cot(u) + \csc(u))$:

$$= -\frac{1}{4} \log(\cot(u) + \csc(u)) + \text{constant}$$

Substitute back for $u = 4x$:

$$= -\frac{1}{4} \log(\cot(2x)) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \frac{1}{4} \log(2\sin(2x)) - \frac{1}{4} \log(2\cos(2x)) + \text{constant}$$

83.

(25) $\int \sqrt[3]{x}\,(1-\sqrt{x})\,dx$

$\int x^{1/3}(1-x^{1/2})\,dx$ • rewrite exponents

$\int (x^{1/3}-x^{5/6})\,dx$ • simplify

$\frac{3}{4}x^{4/3}-\frac{6}{11}x^{11/6}+C$ • integrate

Problem 76

$$\int \frac{e^x}{e^{2x}-1}dx$$

Make the substitution $u = e^x$ and $du = e^x dx$

$$\int \frac{1}{u^2-1}dx$$

The anti-derivative of $\frac{1}{1-u^2}$ is arctanh, which is used to solve the integral

$$-\operatorname{arctanh}(u) + C$$

Substituting e^xback in for u

$$-\operatorname{arctanh}(e^x) + C$$

77

Possible intermediate steps:

$$\int \log(1 + x^2)\, dx$$

For the integrand $\log(x^2 + 1)$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where
$f = \log(x^2 + 1)$, $\quad dg = dx$,
$df = \frac{2x}{x^2+1}\, dx$, $\quad g = x$:

$$= x \log(x^2 + 1) - \int \frac{2x^2}{x^2+1}\, dx$$

Factor out constants:

$$= x \log(x^2 + 1) - 2 \int \frac{x^2}{x^2+1}\, dx$$

For the integrand $\frac{x^2}{x^2+1}$, do long division:

$$= x \log(x^2 + 1) - 2 \int \left(1 - \frac{1}{x^2+1}\right) dx$$

Integrate the sum term by term and factor out constants:

$$= x \log(x^2 + 1) + 2 \int \frac{1}{x^2+1}\, dx - 2 \int 1\, dx$$

The integral of $\frac{1}{x^2+1}$ is $\tan^{-1}(x)$:

$$= x \log(x^2 + 1) + 2 \tan^{-1}(x) - 2 \int 1\, dx$$

The integral of 1 is x:

$$= x \log(x^2 + 1) - 2x + 2 \tan^{-1}(x) + \text{constant}$$

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78) $\int \frac{x^4}{(x^5+1)^3}$

$u = x^5 + 1$

$du = 5x^4\,dx$

$= \frac{1}{5}\int \frac{du}{u^3} = \frac{1}{5}\int u^{-3}\,du$

$= \frac{-1}{10}u^{-2} + C = \frac{-1}{10(x^5+1)^2} + C$

79 $\int x \tan^2 x \, dx$

$= \int x(\sec^2(x) - 1)\,dx$

$= \int (x\sec^2(x) - x)\,dx$

$= \int x\sec^2(x)\,dx - \int x\,dx$

{substitute $f = x$ $\quad dg = \sec^2 x\,dx$
$df = dx$ $\quad g = \tan(x)$ }

$= x\tan x - \int x\,dx - \int \tan x\,dx$

$= x\tan x - \int x\,dx + \log(\cos(x))$

$= \frac{x^2}{2} + x\tan(x) + \log(\cos(x)) + c$

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Page 368, Problem Number Eighty

Integrate: $\int \frac{\tan^{-1} 2x}{1+4x^2} dx$

You should begin this integral by making the simple substitution of u = 2x. You will also find that $du = 2dx$. This leaves you with the integral:

$$\frac{1}{2}\int \frac{\tan^{-1} u}{1+u^2} du = \frac{1}{2}\int \frac{1}{1+u^2} \cdot \tan^{-1} u \, du$$

I separated the arc-tangent from the fraction to make this next step easier to see. At this point you know that your derivative yields an arc-tangent in the numerator multiplied by the denominator – which looks exactly like the derivative of arc-tangent of u. After some deliberation, you should be able to see that the derivative of arc-tangent squared would give you such an answer, so you should integrate the above equation to get:

$$\frac{1}{2} \cdot \frac{1}{2} (\tan^{-1} u)^2 + C = \frac{1}{4} (\tan^{-1} u)^2 + C$$

When you put u back into terms of x you have the final answer of:

$$\boldsymbol{\frac{1}{4} (\tan^{-1} 2x)^2 + C}$$

#81

$$\int \sec^7 x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$\cdots = \int u^6 \, du = \frac{1}{7} u^7 + C$$

$$= \frac{1}{7} \sec^7 x + C$$

82.) $$\int \frac{1}{x^2 + 5x + 6}\, dx = \log(-x - 2) - \log(x + 3) + \text{constant}$$

Log is the natural logarithm

Possible intermediate steps:

$$\int \frac{1}{6 + 5x + x^2}\, dx$$

For the integrand $\frac{1}{x^2 + 5x + 6}$, complete the square:

$$= \int \frac{1}{\left(x + \frac{5}{2}\right)^2 - \frac{1}{4}}\, dx$$

For the integrand $\frac{1}{\left(x + \frac{5}{2}\right)^2 - \frac{1}{4}}$, substitute $u = x + \frac{5}{2}$ and $du = dx$:

$$= \int \frac{1}{u^2 - \frac{1}{4}}\, du$$

The integral of $\frac{1}{u^2 - \frac{1}{4}}$ is $-2\tanh^{-1}(2u)$:

$$= -2\tanh^{-1}(2u) + \text{constant}$$

Substitute back for $u = x + \frac{5}{2}$:

$$= -2\tanh^{-1}(2x + 5) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \log(-x - 2) - \log(x + 3) + \text{constant}$$

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#83

Possible intermediate steps:

$\int x \sin^{-1}(x)\, dx$

For the integrand $x \sin^{-1}(x)$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where
$f = \sin^{-1}(x)$, $dg = x dx$,
$df = \frac{1}{\sqrt{1-x^2}} dx$, $g = \frac{x^2}{2}$:

$= \frac{1}{2} x^2 \sin^{-1}(x) - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}}\, dx$

For the integrand, $\frac{x^2}{\sqrt{1-x^2}}$ substitute $x = \sin(u)$ and $dx = \cos(u)\, du$. Then $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \cos(u)$ and $u = \sin^{-1}(x)$:

$= \frac{1}{2} x^2 \sin^{-1}(x) - \frac{1}{2} \int \sin^2(u)\, du$

Write $\sin^2(u)$ as $\frac{1}{2} - \frac{1}{2}\cos(2u)$:

$= \frac{1}{2} x^2 \sin^{-1}(x) - \frac{1}{2} \int \left(\frac{1}{2} - \frac{1}{2}\cos(2u)\right) du$

Integrate the sum term by term and factor out constants:

$= -\frac{1}{2} \int \frac{1}{2}\, du + \frac{1}{4} \int \cos(2u)\, du + \frac{1}{2} x^2 \sin^{-1}(x)$

For the integrand $\cos(2u)$, substitute $s = 2u$ and $ds = 2du$:

$= \frac{1}{8} \int \cos(s)\, ds - \frac{1}{2} \int \frac{1}{2}\, du + \frac{1}{2} x^2 \sin^{-1}(x)$

The integral of $\frac{1}{2}$ is $\frac{u}{2}$:

$= \frac{1}{8} \int \cos(s)\, ds - \frac{u}{4} + \frac{1}{2} x^2 \sin^{-1}(x)$

The integral of $\cos(s)$ is $\sin(s)$:

$= \frac{\sin(s)}{8} - \frac{u}{4} + \frac{1}{2} x^2 \sin^{-1}(x) + \text{constant}$

Substitute back for $s = 2u$:

$= -\frac{u}{4} + \frac{1}{8} \sin(2u) + \frac{1}{2} x^2 \sin^{-1}(x) + \text{constant}$

Substitute back for $u = \sin^{-1}(x)$:

$= \frac{1}{4} \sqrt{1-x^2}\, x + \frac{1}{2} x^2 \sin^{-1}(x) - \frac{1}{4} \sin^{-1}(x) + \text{constant}$

Which is equal to:

$= \frac{1}{4} \left(\sqrt{1-x^2}\, x + (2x^2 - 1) \sin^{-1}(x)\right) + \text{constant}$

92.

(84) $\int \frac{1+\cos^2 x}{1-\cos^2 x}\,dx$

$\int \frac{1+\cos^2 x}{\sin^2 x}\,dx$ • change $(1-\cos^2 x)$ to $(\sin^2 x)$

$\int \frac{1}{\sin^2 x}\,dx + \int \frac{\cos^2 x}{\sin^2 x}\,dx$ • separate terms

$\int \csc^2 x\,dx + \int \cot^2 x\,dx$ • simplify

$-\cot x - x - \cot x + C$ • integrate

$-2\cot x - x + C$ • simplify

Problem 85

$$\int \frac{x^3}{1+x^8}\,dx$$

Start by making the substitution u = x^4 and du = $4x^3$

$$\frac{1}{4}\int \frac{1}{1+u^2}$$

The integral of $\frac{1}{1+u^2}$ is arctan(u)

$$\frac{\arctan(u)}{4} + C$$

Substitute x^4 back in for u

$$\frac{\arctan(x^4)}{4} + C$$

86

Possible intermediate steps:

$$\int \frac{\tan^{-1}(x)}{x^2}\,dx$$

For the integrand $\frac{\tan^{-1}(x)}{x^2}$, integrate by parts, $\int f\,dg = f g - \int g\,df$, where

$f = \tan^{-1}(x), \quad dg = \frac{1}{x^2}\,dx,$

$df = \frac{1}{x^2+1}\,dx, \quad g = -\frac{1}{x}:$

$$= -\int -\frac{1}{x(x^2+1)}\,dx - \frac{\tan^{-1}(x)}{x}$$

Factor out constants:

$$= \int \frac{1}{x(x^2+1)}\,dx - \frac{\tan^{-1}(x)}{x}$$

For the integrand $\frac{1}{x(x^2+1)}$, substitute $u = x^2$ and $du = 2x\,dx$:

$$= \frac{1}{2}\int \frac{1}{u(u+1)}\,du - \frac{\tan^{-1}(x)}{x}$$

For the integrand $\frac{1}{u(u+1)}$, use partial fractions:

$$= \frac{1}{2}\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du - \frac{\tan^{-1}(x)}{x}$$

Integrate the sum term by term and factor out constants:

$$= \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int \frac{1}{u+1}\,du - \frac{\tan^{-1}(x)}{x}$$

For the integrand $\frac{1}{u+1}$, substitute $s = u+1$ and $ds = du$:

$$= -\frac{1}{2}\int \frac{1}{s}\,ds + \frac{1}{2}\int \frac{1}{u}\,du - \frac{\tan^{-1}(x)}{x}$$

The integral of $\frac{1}{s}$ is $\log(s)$:

$$= -\frac{\log(s)}{2} + \frac{1}{2}\int \frac{1}{u}\,du - \frac{\tan^{-1}(x)}{x}$$

The integral of $\frac{1}{u}$ is $\log(u)$:

$$= -\frac{\log(s)}{2} + \frac{\log(u)}{2} - \frac{\tan^{-1}(x)}{x} + \text{constant}$$

Substitute back for $s = u+1$:

$$= \frac{1}{2}\left(\log(u) - \log(u+1) - \frac{2\tan^{-1}(x)}{x}\right) + \text{constant}$$

Substitute back for $u = x^2$:

$$= \frac{1}{2}\left(\log(x^2) - \log(x^2+1) - \frac{2\tan^{-1}(x)}{x}\right) + \text{constant}$$

An alternative form of the integral is:

$$= \frac{1}{2}\left(\log\left(\frac{x^2}{x^2+1}\right) - \frac{2\tan^{-1}(x)}{x}\right) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= -\frac{1}{2}\log(x^2+1) + \log(x) - \frac{\tan^{-1}(x)}{x} + \text{constant}$$

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87) $$\int \frac{\sec^2 x}{\sqrt{\sec^2 x - 1}}\,dx$$

$$= \int \frac{\sec^2 x}{\tan x}\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int \frac{du}{u} = \ln(u) + C$$

$$= \ln(\tan x) + C$$

88

$$\int \frac{dx}{1+2e^x-e^{-x}} \quad \left\{ \text{substitute: } \begin{array}{l} u=e^x \\ du=e^x dx \end{array} \right\}$$

$$= \int \frac{1}{2u^2+u-1}\, du$$

$$= \int \frac{1}{\left(\sqrt{2}u+\frac{1}{2\sqrt{2}}\right)^2-\frac{9}{8}}\, du \quad \left\{ \text{substitute: } \begin{array}{l} s=\sqrt{2}u+\frac{1}{2\sqrt{2}} \\ ds=\sqrt{2}\,du \end{array} \right\}$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{s^2-\frac{9}{8}}\, ds$$

$$= \frac{-2}{3} \tanh^{-1}\left(\frac{2\sqrt{2}s}{3}\right)+C \quad \left\{ \text{substitute } s=\sqrt{2}u+\frac{1}{2\sqrt{2}} \right\}$$

$$= \frac{-2}{3} \tanh^{-1}\left(\frac{4u}{3}+\frac{1}{3}\right)+C \quad \left\{ \text{substitute } u=e^x \right\}$$

$$\boxed{= \frac{-2}{3} \tanh^{-1}\left(\frac{4e^x}{3}+\frac{1}{3}\right)+C}$$

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Page 368, Problem Number Eighty-Nine

Integrate: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

To integrate this equation, just make the u substitution of $u = \sqrt{x}$ and find that $du = \frac{1}{2\sqrt{x}} dx$. You should end up with:

$$2 \int e^u \, du$$

Which is simply:

$$2e^{\sqrt{x}} + c$$

#90 $\int \frac{\sqrt{x^2+9}}{x}\,dx$

Let $u^2 = x^2+9$

$2u\,du = 2x\,dx$

$\frac{dx}{x} = \frac{u\,du}{x^2} = \frac{u\,du}{u^2-9}$

$$\ldots = \int \frac{u\sqrt{u^2}}{u^2-9}\,du = \int \frac{u^2}{u^2-9}\,du$$

$$= \int \frac{u^2-9+9}{u^2-9}\,du = u + 9\int \frac{1}{u^2-9}\,du$$

Partial fractions

$$\frac{1}{u^2-9} = \frac{A}{u-3} + \frac{B}{u+3}$$

$$1 = (u+3)A + (u-3)B$$

at $x=-3$ $\quad 1 = -6B \quad B = -1/6$

$x=3$ $\quad 1 = 6A \quad A = 1/6$

$$\ldots = u + \frac{9}{6}\ln|u-3| - \frac{9}{6}\ln|u+3| + C$$

$$= \sqrt{x^2+9} + \frac{3}{2}\ln(\sqrt{x^2+9}-3) - \frac{3}{2}\ln(\sqrt{x^2+9}+3) + C$$

91.) $$\int \frac{\log(x^{10})}{x}\,dx = \frac{1}{20}\log^2(x^{10}) + \text{constant}$$

Log is the natural logarithm

Possible intermediate steps:

$$\int \frac{\log(x^{10})}{x}\,dx$$

For the integrand $\frac{\log(x^{10})}{x}$, substitute $u = \log(x^{10})$ and $du = \frac{10}{x}\,dx$:

$$= \frac{1}{10}\int u\,du$$

The integral of u is $\frac{u^2}{2}$:

$$= \frac{u^2}{20} + \text{constant}$$

Substitute back for $u = \log(x^{10})$:

$$= \frac{1}{20}\log^2(x^{10}) + \text{constant}$$

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#92

Possible intermediate steps:

$$\int \sec(5x)\,dx$$

For the integrand $\sec(5x)$, substitute $u = 5x$ and $du = 5dx$:

$$= \frac{1}{5}\int \sec(u)\,du$$

The integral of $\sec(u)$ is $\log(\sec(u) + \tan(u))$:

$$= \frac{1}{5}\log(\tan(u) + \sec(u)) + \text{constant}$$

Substitute back for $u = 5x$:

$$= \frac{1}{5}\log(\tan(5x) + \sec(5x)) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \frac{1}{5}\log\left(\sin\left(\frac{5x}{2}\right) + \cos\left(\frac{5x}{2}\right)\right) - \frac{1}{5}\log\left(\cos\left(\frac{5x}{2}\right) - \sin\left(\frac{5x}{2}\right)\right) + \text{constant}$$

(93) $\int \frac{\sqrt{x-2}}{x+2}\,dx$

$\int \left(\frac{1}{\sqrt{x-2}}\right)\left(\frac{x-2}{x+2}\right)dx$	• separate $\left(\frac{1}{\sqrt{x-2}}\right)$
$\int \left(\frac{1}{\sqrt{x-2}}\right)\left(\frac{x+2}{x+2} - \frac{4}{x+2}\right)dx$	• rearrange to make $\left(\frac{x+2}{x+2}\right)$
$\int (x-2)^{-1/2}\left(1 - \frac{1}{1/4(x+2)}\right)dx$	• simplify
$\int \left[(x-2)^{-1/2} - \left(\frac{1}{1+1/4(x-2)}\right)(x-2)^{-1/2}\right]dx$	• multiply across and rearrange
$\int \left[(x-2)^{-1/2} - \left(\frac{1}{1+(1/2\sqrt{x-2})^2}\right)(x-2)^{-1/2}\right]dx$	• change $[1/4(x-2)]$ to $(1/2\sqrt{x-2})^2$

$u = 1/2\sqrt{x-2} \qquad du = 1/4(x-2)^{-1/2}dx$

$\int (x-2)^{-1/2}dx - 4\int \frac{1}{1+u^2}du$	• use a u-substitution
$2\sqrt{x-2} - 4\tan^{-1}(u) + C$	• integrate
$2\sqrt{x-2} - 4\tan^{-1}(1/2\sqrt{x-2}) + C$	• put in terms of x

Problem 94

$$\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$$

Start by rationalizing the denominator by multiplying both sides with the conjugate

$$\int \frac{1}{\sqrt{x}+\sqrt{x+1}} * \frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}\,dx = \int (\sqrt{x+1}-\sqrt{x})dx$$

Split up the main integral into two smaller integrals

$$\int \sqrt{x+1}\,dx - \int \sqrt{x}\,dx$$

Make the substitution u = x + 1 and du = dx

$$\int \sqrt{u}\,du - \int \sqrt{x}\,dx$$

The anti-derivative of $\sqrt{x}$ is $\frac{2x^{\frac{3}{2}}}{3}$ and the anti-derivative of $\sqrt{u}$ is $\frac{2u^{\frac{3}{2}}}{3}$

$$\frac{2u^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{3} + C$$

Substitute x + 1 back in for u

$$\frac{2(x+1)^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{3} + C$$

95

$$\int \frac{1}{\sin^2(x)}\, dx = -\cot(x) + \text{constant}$$

Possible intermediate steps:

$$\int \csc^2(x)\, dx$$

The integral of $\csc^2(x)$ is $-\cot(x)$:

$$= -\cot(x) + \text{constant}$$

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96.) $\int \frac{1}{\sqrt{9x^2 + 12x - 5}}\, dx$

$= \int \frac{1}{\sqrt{9\left(x + \frac{6}{9}\right)^2 - \frac{36}{9} - 5}}\, dx$

$= \int \frac{1}{\sqrt{9\left(x + \frac{6}{9}\right)^2 - 9}}\, dx$

$= \frac{1}{3} \int \frac{1}{\sqrt{u^2 - 1}}$ $\qquad u = x + \frac{6}{9}$

$u = \cosh\theta$
$du = \sinh\theta \, d\theta$

$= \frac{1}{3} \int \frac{\sinh\theta}{\sqrt{\sinh\theta}}\, d\theta = \frac{1}{3} \int d\theta$

$= \frac{1}{3}\theta$

$= \frac{1}{3} \cosh^{-1} u$

$= \frac{1}{3} \cosh^{-1}\left(x + \frac{6}{9}\right) + C$

97

$$\int \sqrt{(1+3x)(1-3x)}\,dx$$

$$= \int \sqrt{1-9x^2}\,dx \quad \left\{ \text{substitute: } x = \frac{\sin(u)}{3},\ dx = \frac{1}{3}\cos(u)\,du \right\}$$

$$= \frac{1}{3}\int \cos^2(u)\,du$$

$$= \frac{1}{3}\int \left(\frac{1}{2}\cos(2u) + \frac{1}{2}\right) du$$

$$= \frac{1}{3}\int \frac{1}{2}\,du + \frac{1}{6}\int \cos(2u)\,du \quad \left\{ \text{substitute } 2u = s,\ 2du = ds \right\}$$

$$= \frac{1}{12}\int \cos(s)\,ds + \frac{1}{3}\int \frac{1}{2}\,du$$

$$= \frac{1}{12}\int \cos(s)\,ds = \frac{u}{6}$$

$$= \frac{\sin(s)}{12} + \frac{u}{6} + c \quad \left\{ \text{substitute } s = 2u \right\}$$

$$= \frac{u}{6} + \frac{1}{6}\sin(u)\cos(u) + c \quad \left\{ \text{substitute } u = \sin^{-1}(3x) \right\}$$

$$\boxed{= \frac{1}{2}\sqrt{1-9x^2}\;x + \frac{1}{6}\sin^{-1}(3x) + c}$$

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Page 368, Problem Number Ninety-Eight

Integrate: $\int \sin^3 x \cos^2 x\, dx$

To begin this problem you should make the u substitution of $u = \cos x$. This means that your $du = -\sin x\, dx$. Using trigonometric identities and by plugging in your u you should get this answer. (I have shown the steps):

$$\int \mathrm{Sin}^2 x \cos^2 x \sin x\, dx = \int (1 - \cos^2 \mathrm{x}) \cos^2 x \sin x\, dx =$$

$$-\int (1 - \mathrm{u}^2)\, \mathrm{u}^2\, du = -\int \mathrm{u}^2 - \mathrm{u}^4\, du$$

Integrate that final step to get:

$$\frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

Put the equation back in terms of x to get your final answer of:

$$\boldsymbol{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$$

#99 $\int \frac{2x+5}{x^2+5x+6}\,dx$

$u = x^2+5x+6$

$du = (2x+5)\,dx$

$\therefore = \int \frac{du}{u} = \ln|u| + c$

$= \ln|x^2+5x+6| + C$

100.) $\int \sin^2(x)\cos^3(x)\,dx = \frac{\sin(x)}{8} - \frac{1}{48}\sin(3x) - \frac{1}{80}\sin(5x) + \text{constant}$

Possible intermediate steps:

$\int \cos^3(x)\sin^2(x)\,dx$

For the integrand $\cos^3(x)\sin^2(x)$, use the trigonometric identity $\sin^2(x) = 1 - \cos^2(x)$:

$= \int \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)\,dx$

For the integrand $\cos(x)\sin^2(x)\left(1 - \sin^2(x)\right)$, substitute $u = \sin(x)$ and $du = \cos(x)dx$:

$= \int u^2\left(1 - u^2\right)du$

For the integrand $u^2\left(1 - u^2\right)$, do long division:

$= \int \left(u^2 - u^4\right)du$

Integrate the sum term by term and factor out constants:

$= \int u^2\,du - \int u^4\,du$

The integral of u^4 is $\frac{u^5}{5}$:

$= \int u^2\,du - \frac{u^5}{5}$

The integral of u^2 is $\frac{u^3}{3}$:

$= \frac{u^3}{3} - \frac{u^5}{5} + \text{constant}$

Substitute back for $u = \sin(x)$:

$= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + \text{constant}$

Which is equal to:

$= \frac{\sin(x)}{8} - \frac{1}{48}\sin(3x) - \frac{1}{80}\sin(5x) + \text{constant}$

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Possible intermediate steps:

101) $$\int \frac{x}{-3-2x^2+x^4}\,dx$$

For the integrand $\frac{x}{x^4-2x^2-3}$, substitute $u = x^2$ and $du = 2x\,dx$:

$$= \frac{1}{2}\int \frac{1}{u^2-2u-3}\,du$$

For the integrand $\frac{1}{u^2-2u-3}$, complete the square:

$$= \frac{1}{2}\int \frac{1}{(u-1)^2-4}\,du$$

For the integrand $\frac{1}{(u-1)^2-4}$, substitute $s = u-1$ and $ds = du$:

$$= \frac{1}{2}\int \frac{1}{s^2-4}\,ds$$

The integral of $\frac{1}{s^2-4}$ is $-\frac{1}{2}\tanh^{-1}\left(\frac{s}{2}\right)$:

$$= -\frac{1}{4}\tanh^{-1}\left(\frac{s}{2}\right) + \text{constant}$$

Substitute back for $s = u-1$:

$$= -\frac{1}{4}\tanh^{-1}\left(\frac{u-1}{2}\right) + \text{constant}$$

Substitute back for $u = x^2$:

$$= -\frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\left(x^2-1\right)\right) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= \frac{1}{8}\log\left(3-x^2\right) - \frac{1}{8}\log\left(x^2+1\right) + \text{constant}$$

[redacted].

(102) $\int \sqrt{1+\sqrt{1+\sqrt{x}}}\,dx$

$x = (u^2-1)^2 \qquad dx = 4(u^3-u)\,du$

$4\int \sqrt{1+u}\,(u^3-u)\,du$ • use u-substitution

$u = v-1 \qquad du = dv$

$4\int \sqrt{v}\,[(v-1)^3 - v + 1]\,dv$ • use v-substitution

$4\int v^{1/2}(v^3-3v^2+2v)\,dv$ • simplify

$4\int (v^{7/2} - 3v^{5/2} + 2v^{3/2})\,dv$ • multiply out

$4\left(\frac{2}{9}v^{9/2} - \frac{6}{7}v^{7/2} + \frac{4}{5}v^{5/2}\right) + C$ • integrate

$\frac{8}{9}v^{9/2} - \frac{24}{7}v^{7/2} + \frac{16}{5}v^{5/2} + C$ • simplify

$\frac{8}{9}(1+u)^{9/2} - \frac{24}{7}(1+u)^{7/2} + \frac{16}{5}(1+u)^{5/2} + C$ • put in terms of u

$\frac{8}{9}\left(1+\sqrt{1+\sqrt{x}}\right)^{9/2} - \frac{24}{7}\left(1+\sqrt{1+\sqrt{x}}\right)^{7/2} + \frac{16}{5}\left(1+\sqrt{1+\sqrt{x}}\right)^{5/2} + C$ • put in terms of x

Problem 103

$$\int \frac{x^4}{x^3-1}\,dx$$

Begin by modifying the integral using long division and split into three smaller integrals

$$\int (\frac{1-x}{3(x^3+x+1)} + x + \frac{1}{3(x-1)})\,dx = \frac{1}{3}\int \frac{1-x}{x^2+\ x+1}dx + \frac{1}{3}\int \frac{1}{x-1}dx + \int xdx$$

Use algebra to split the integral $\frac{1-x}{x^2+x+1}$ into two fractions and split into two smaller integrals

$$\frac{1}{3}\int \left(\frac{3}{2(x^2+\ x+1)} - \frac{2x+1}{2(x^2+\ x+1)}\right)dx = \ \frac{1}{2}\int \frac{1}{x^2+\ x+1}dx - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}d$$

For the second integral, substitute u = x^2 + x + 1 and du = 2x + 1 dx

$$\frac{1}{2}\int \frac{1}{x^2+\ x+1}dx - \frac{1}{6}\int \frac{1}{u}du + \frac{1}{3}\int \frac{1}{x-1}dx + \int xdx$$

Complete the square for the first integral

$$\frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx - \frac{1}{6}\int \frac{1}{u}du + \frac{1}{3}\int \frac{1}{x-1}dx + \int xdx$$

For the first integral, substitute w = x + $\frac{1}{2}$ and dw = dx and in the third substitute v = x—1 and dv = dx

$$\frac{1}{2}\int \frac{1}{w^2+\frac{3}{4}}dw - \frac{1}{6}\int \frac{1}{u}du + \frac{1}{3}\int \frac{1}{v}dv + \int xdx$$

The first integral comes out to $\frac{\arctan\left(\frac{2w}{\sqrt{3}}\right)}{\sqrt{3}}$, the second ln(u), the third ln(v), and the fourth $\frac{x^2}{2}$

$$\frac{\arctan\left(\frac{2w}{\sqrt{3}}\right)}{\sqrt{3}} - \frac{\ln(u)}{6} + \frac{\ln(v)}{3} + \frac{x^2}{2} + C$$

Substitute back in x - 1 for v, x + $\frac{1}{2}$ for w, and x^2 + x + 1 for u

$$\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}} - \frac{\ln(\mathrm{x2\ +\ x\ +\ 1})}{6} + \frac{\ln(x-1)}{3} + \frac{x^2}{2} + C$$

104

$$\int x \sec^2(x)\, dx = x \tan(x) + \log(\cos(x)) + \text{constant}$$

Possible intermediate steps:

$$\int x \sec^2(x)\, dx$$

For the integrand $x \sec^2(x)$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where
$f = x, \quad dg = \sec^2(x) dx,$
$df = dx, \quad g = \tan(x)$:

$$= x \tan(x) - \int \tan(x)\, dx$$

The integral of $\tan(x)$ is $-\log(\cos(x))$:

$$= x \tan(x) + \log(\cos(x)) + \text{constant}$$

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105) $\int \frac{\sin^3 x}{\cos^5 x}\, dx$

$= \int \frac{(1-\cos^2 x)(\sin x)}{\cos^5 x}\, dx$ $\quad u = \cos x$, $du = -\sin x\, dx$

$= \int -\frac{1-u^2}{u^5}\, du$

$= \int -\left(\frac{1}{u^5} - \frac{1}{u^3}\right) du$

$= \int -u^{-5} + u^{-3}\, du$

$= \frac{1}{4}u^{-4} - \frac{1}{2}u^{-2} + C$

$= \frac{1}{4\cos^4 x} - \frac{1}{2\cos^2 x} + C$

106

$$\int \sin 2x \cos x \, dx$$

$$= \frac{1}{2} \int (\sin(x) + \sin(3x))\, dx$$

~~substitute~~

$$= \frac{1}{2} \int \sin(x)\, dx + \frac{1}{2} \int \sin(3x)\, dx$$

$$\left\{ \text{substitute: } u = 3x, \quad du = 3dx \right\}$$

$$= \frac{1}{6} \int \sin(u)\, du + \frac{1}{2} \int \sin(x)\, dx$$

$$= \frac{1}{2} \int \sin(x)\, dx - \frac{\cos(u)}{6}$$

$$= -\frac{\cos(u)}{6} - \frac{\cos x}{2} + C$$

$$\left\{ \text{substitute } u = 3x \right\}$$

$$\boxed{= -\frac{\cos(x)}{2} - \frac{1}{6}\cos(3x) + C}$$

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Page 368, Problem Number One-Hundred and Seven

Integrate: $\int sinx \cdot cos2x\, dx$

To make your life easier, begin this integral by rearranging the equation into something that is easier to integrate. Use the Pythagorean identity and the angle addition formulas to get:

$$\int sinx \cdot (2\cos^2 x - 1)\, dx$$

You should now make a u substitution of $u = \cos x$. You will find that $du = -\sin x\, dx$. You should end up with:

$$-\int (2u^2 - 1)\, du$$

Integrate this and put it in terms of x to get:

$$-\frac{2}{3}u^3 + u + C = \cos x - \frac{2}{3}\cos^3 x + C$$

#108 $\displaystyle\int \frac{4}{x^4-1}\,dx = 4\int \frac{1}{(x-1)(x+1)(x^2+1)}\,dx$

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$1 = (x+1)(x^2+1)A + (x-1)(x^2+1)B + (x^2-1)(Cx+D)$$

at $x=1$ $\quad 1 = 4A \quad A = 1/4$

at $x=-1$ $\quad 1 = -4B \quad B = -1/4$

at $x=0$ $\quad 1 = A - B - D = \frac{1}{4} + \frac{1}{4} - D \quad D = -1/2$

equating powers of x^3:

$$0 = x^3A + x^3B + x^3C = x^3C \qquad C = 0$$

$$\text{''} = \int \left(\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} \right) dx$$

$$= \ln|x-1| - \ln|x+1| - 2\arctan x + C$$

109.) $$\int \frac{x}{\sqrt[3]{x-1}}\,dx = \frac{3}{10}\,(x-1)^{2/3}\,(2\,x+3) + \text{constant}$$

Possible intermediate steps:

$$\int \frac{x}{\sqrt[3]{-1+x}}\,dx$$

For the integrand $\frac{x}{\sqrt[3]{x-1}}$, substitute $u = x-1$ and $du = dx$:

$$= \int \frac{u+1}{\sqrt[3]{u}}\,du$$

Expanding the integrand $\frac{u+1}{\sqrt[3]{u}}$ gives $u^{2/3} + \frac{1}{\sqrt[3]{u}}$:

$$= \int \left(u^{2/3} + \frac{1}{\sqrt[3]{u}}\right) du$$

Integrate the sum term by term:

$$= \int u^{2/3}\,du + \int \frac{1}{\sqrt[3]{u}}\,du$$

The integral of $\frac{1}{\sqrt[3]{u}}$ is $\frac{3\,u^{2/3}}{2}$:

$$= \frac{3\,u^{2/3}}{2} + \int u^{2/3}\,du$$

The integral of $u^{2/3}$ is $\frac{3\,u^{5/3}}{5}$:

$$= \frac{3\,u^{5/3}}{5} + \frac{3\,u^{2/3}}{2} + \text{constant}$$

Substitute back for $u = x-1$:

$$= \frac{3}{5}\,(x-1)^{5/3} + \frac{3}{2}\,(x-1)^{2/3} + \text{constant}$$

Which is equal to:

$$= \frac{3}{10}\,(x-1)^{2/3}\,(2\,x+3) + \text{constant}$$

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#110:

Possible intermediate steps:

$$\int \frac{1}{5+4e^{-x}+e^{x}}\,dx$$

For the integrand $\frac{1}{5+4e^{-x}+e^{x}}$, substitute $u=e^{x}$ and $du=e^{x}dx$:

$$=\int \frac{1}{u^2+5u+4}\,du$$

For the integrand $\frac{1}{u^2+5u+4}$, complete the square:

$$=\int \frac{1}{\left(u+\frac{5}{2}\right)^2-\frac{9}{4}}\,du$$

For the integrand $\frac{1}{\left(u+\frac{5}{2}\right)^2-\frac{9}{4}}$, substitute $s=u+\frac{5}{2}$ and $ds=du$:

$$=\int \frac{1}{s^2-\frac{9}{4}}\,ds$$

The integral of $\frac{1}{s^2-\frac{9}{4}}$ is $-\frac{2}{3}\tanh^{-1}\left(\frac{2s}{3}\right)$:

$$=-\frac{2}{3}\tanh^{-1}\left(\frac{2s}{3}\right)+\text{constant}$$

Substitute back for $s=u+\frac{5}{2}$:

$$=-\frac{2}{3}\tanh^{-1}\left(\frac{2u}{3}+\frac{5}{3}\right)+\text{constant}$$

Substitute back for $u=e^{x}$:

$$=-\frac{2}{3}\tanh^{-1}\left(\frac{2e^{x}}{3}+\frac{5}{3}\right)+\text{constant}$$

~~119~~.

(111) $\int \ln(ax+b)\,dx$

$u = ax+b \qquad du = a\,dx$

$\frac{1}{a}\int \ln u\,du$ • use u-substitution

$\frac{1}{a}(u\ln u - u) + c$ • integrate

$\frac{1}{a}\left[(ax+b)\ln(ax+b) - (ax+b)\right] + c$ • put in terms of x

$\frac{1}{a}(ax+b)\ln(ax+b) - x - \frac{b}{a} + c$ • simplify

$\frac{1}{a}(ax+b)\ln(ax+b) - x + c$ • $\left(\frac{b}{a}\right)$, a constant, can be included in (c)

Problem 112

$$\int \frac{\cos^4 x}{\sin^2 x} dx$$

Start by substituting $\cot^2$ for $\frac{cos^2 x}{\sin^2 x}$

$$\int \cos^2 x \cot^2 x \, dx$$

Use the trigonometric identity $\cot^2 = \csc^2(x) - 1$

$$\int \cos^2 x \, (csc^2 x - 1) \, dx$$

Multiply out the terms and use the trigonometric identity $\cot^2 = \frac{\cos^2 x}{\sin^2 x}$

$$\int (cot^2 x - \cos^2 x) dx$$

Split up the integral into two smaller integrals

$$\int \cot^2 x \, dx - \int \cos^2 x \, dx$$

Rewrite $\cos^2 x \; as \; \frac{\cos(2x)+1}{2}$

$$\int \cot^2 x \, dx - \int \frac{\cos(2\mathrm{x}) + 1}{2} \, dx$$

Split up the integral again into two smaller integrals

$$\int \cot^2 x \, dx - \frac{1}{2} \int cos(2x) \, dx - \int \frac{1}{2} dx$$

In the middle integral, make the substitution u = 2x and du = 2dx

$$\int \cot^2 x \, dx - \frac{1}{4} \int cos(u) \, dx - \int \frac{1}{2} dx$$

The integral of cos(u) is sin(u), the integral of $\frac{1}{2}$ is $\frac{x}{2}$, and the integral of $\cot^2(x)$ is $-x - \cot(x)$

$$-x - \cot(x) - \frac{\sin(u)}{4} - \frac{x}{2} + C$$

Substitute 2x back in for u, and combine like terms

$$-\frac{\sin(2x)}{4} - \frac{3x}{2} - \cot(x) + C$$

113

$$\int e^x \cos(e^x)\, dx = \sin(e^x) + \text{constant}$$

Possible intermediate steps:

$$\int e^x \cos(e^x)\, dx$$

For the integrand $e^x \cos(e^x)$, substitute $u = e^x$ and $du = e^x dx$:

$$= \int \cos(u)\, du$$

The integral of $\cos(u)$ is $\sin(u)$:

$$= \sin(u) + \text{constant}$$

Substitute back for $u = e^x$:

$$= \sin(e^x) + \text{constant}$$

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114) $\int \frac{1}{3x^2 - 13x + 4}\,dx$

$3x^2 - 13x + 4$

$3\left(x^2 - \frac{13}{3}x\right) + 4$

$3\left(x - \frac{13}{6}x\right)^2 - 3\left(\frac{13}{6}\right)^2 + 4$

$= \int \frac{1}{3\left(x - \frac{13}{6}x\right)^2 - \frac{121}{12}}$ $\quad u = x - \frac{13}{6}$

$= \frac{1}{3}\int \frac{1}{u^2 - \frac{121}{12}}\,du$

$= \frac{1}{3} \cdot \frac{1}{\sqrt{\frac{121}{12}}} \int \frac{1}{u^2 - 1}\,du$

$= \frac{2\sqrt{3}}{33}\int \frac{1}{(u-1)(u+1)} = \frac{2\sqrt{3}}{33} \cdot \frac{1}{2}\int\left(\frac{1}{u-1} + \frac{1}{u+1}\right)dx$

$= \frac{-\sqrt{3}}{33}\left(\ln(u-1) + \ln(u+1)\right) + C$

$= \frac{-\sqrt{3}}{33}\left(\ln\left(x - \frac{19}{6}\right) + \ln\left(x - \frac{7}{6}\right)\right) + C$

115

$$\int \sqrt{x}\ \ln x\ dx$$

Substitute: $f = \ln x$ $\quad dg = \sqrt{x}\,dx$

$df = \frac{1}{x}dx$ $\quad g = \frac{2x^{3/2}}{3}$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int \sqrt{x}\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4x^{\frac{3}{2}}}{9} + C$$

$$= \frac{2}{9}x^{\frac{3}{2}}(3\ln x - 2) + C$$

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Page 368, Problem Number One-Hundred and Sixteen

Integrate: $\int \frac{\cos x}{\sin^2(x) - 2\sin(x)+3} dx$

Begin this integral by making the u substitution of $u = sinx$. $du = \cos xdx$ so you should get:

$$\int \frac{du}{u^2-2u+3}$$

Complete the square in the denominator to make your second substitution (v) so that you can eliminate the second term in that quadratic equation. You will end up with $v = u - 1$ and $dv = du$. Here is the equation once everything is substituted in:

$$\int \frac{dv}{v^2+2}$$

Factor out a $\frac{1}{2}$ to get it in the form of the derivative of $\tan^{-1} t$. You should get:

$$\frac{1}{2} \int \frac{dv}{\frac{v^2}{2}+1}$$

You should integrate the equation and put v in terms of x to get:

$$\frac{1}{\sqrt{2}} \tan^{-1}(\frac{v}{\sqrt{2}}) + C = \frac{1}{\sqrt{2}} \tan^{-1}(\frac{\sin x-1}{\sqrt{2}}) + C$$

#117 $\int(\tan x+\cot x)^2\,dx = \int\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)^2 dx$

$= \int\left(\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}\right)^2 dx = \int\left(\frac{1}{\cos x\sin x}\right)^2 dx$

$= \int\left(\frac{2}{\sin 2x}\right)^2 dx = 4\int \csc^2 2x\,dx$

Let $u = 2x$
$du = 2dx$

$\therefore = 2\int \csc^2 u\,du = -2\cot u + C$

$= -2\cot 2x + C$

118.)

$$\int \log(x\sqrt{x})\,dx = x\log(x^{3/2}) - \frac{3x}{2} + \text{constant}$$

Log is the natural logarithm

Possible intermediate steps:

$$\int \log(x^{3/2})\,dx$$

For the integrand $\log(x^{3/2})$, substitute $u = x^{3/2}$ and $du = \frac{3\sqrt{x}}{2}\,dx$:

$$= \frac{2}{3}\int \frac{\log(u)}{\sqrt[3]{u}}\,du$$

For the integrand $\frac{\log(u)}{\sqrt[3]{u}}$, integrate by parts, $\int f\,dg = f\,g - \int g\,df$, where

$f = \log(u), \quad dg = \frac{1}{\sqrt[3]{u}}\,du,$

$df = \frac{1}{u}\,du, \quad g = \frac{3u^{2/3}}{2}$:

$$= u^{2/3}\log(u) - \int \frac{1}{\sqrt[3]{u}}\,du$$

The integral of $\frac{1}{\sqrt[3]{u}}$ is $\frac{3u^{2/3}}{2}$:

$$= u^{2/3}\log(u) - \frac{3u^{2/3}}{2} + \text{constant}$$

Substitute back for $u = x^{3/2}$:

$$= (x^{3/2})^{2/3}\log(x^{3/2}) - \frac{3}{2}(x^{3/2})^{2/3} + \text{constant}$$

Factor the answer a different way:

$$= \frac{1}{2}(x^{3/2})^{2/3}\left(2\log(x^{3/2}) - 3\right) + \text{constant}$$

Which is equivalent for restricted x values to:

$$= x\log(x^{3/2}) - \frac{3x}{2} + \text{constant}$$

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#119.

Possible intermediate steps:

$$\int \frac{80+32x}{(-1+x)(3+x)^2}\,dx$$

For the integrand $\frac{32x+80}{(x-1)(x+3)^2}$, use partial fractions:

$$= \int \left(-\frac{7}{x+3} + \frac{4}{(x+3)^2} + \frac{7}{x-1}\right) dx$$

Integrate the sum term by term and factor out constants:

$$= 7\int \frac{1}{x-1}\,dx + 4\int \frac{1}{(x+3)^2}\,dx - 7\int \frac{1}{x+3}\,dx$$

For the integrand $\frac{1}{x-1}$, substitute $u = x-1$ and $du = dx$:

$$= 7\int \frac{1}{u}\,du + 4\int \frac{1}{(x+3)^2}\,dx - 7\int \frac{1}{x+3}\,dx$$

For the integrand $\frac{1}{(x+3)^2}$, substitute $s = x+3$ and $ds = dx$:

$$= 4\int \frac{1}{s^2}\,ds + 7\int \frac{1}{u}\,du - 7\int \frac{1}{x+3}\,dx$$

For the integrand $\frac{1}{x+3}$, substitute $p = x+3$ and $dp = dx$:

$$= -7\int \frac{1}{p}\,dp + 4\int \frac{1}{s^2}\,ds + 7\int \frac{1}{u}\,du$$

The integral of $\frac{1}{p}$ is $\log(p)$:

$$= -7\log(p) + 4\int \frac{1}{s^2}\,ds + 7\int \frac{1}{u}\,du$$

The integral of $\frac{1}{s^2}$ is $-\frac{1}{s}$:

$$= -7\log(p) - \frac{4}{s} + 7\int \frac{1}{u}\,du$$

The integral of $\frac{1}{u}$ is $\log(u)$:

$$= -7\log(p) - \frac{4}{s} + 7\log(u) + \text{constant}$$

Substitute back for $p = x+3$:

$$= -\frac{4}{s} + 7\log(u) - 7\log(x+3) + \text{constant}$$

Substitute back for $s = x+3$:

$$= \frac{7(x+3)(\log(u) - \log(x+3)) - 4}{x+3} + \text{constant}$$

Substitute back for $u = x-1$:

$$= \frac{7(x+3)(\log(x-1) - \log(x+3)) - 4}{x+3} + \text{constant}$$

Which is equal to:

$$= 16\left(-\frac{1}{4(x+3)} + \frac{7}{16}\log(x-1) - \frac{7}{16}\log(x+3)\right) + \text{constant}$$

(120) $\int \frac{1}{a^2+b^2x^2}\,dx$

$\int \frac{1}{1+\frac{b^2}{a^2}x^2} \cdot \frac{1}{a^2}\,dx$ • separate $\left(\frac{1}{a^2}\right)$

$\frac{1}{a^2}\int \frac{1}{1+\left(\frac{b}{a}x\right)^2}\,dx$ • change $\left(\frac{b^2}{a^2}x^2\right)$ to $\left(\frac{b}{a}x\right)^2$

$u=\frac{b}{a}x \qquad du=\frac{b}{a}\,dx$

$\frac{1}{ab}\int \frac{1}{1+u^2}\,du$ • use a u-substitution

$\frac{1}{ab}\tan^{-1}(u)+C$ • integrate

$\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right)+C$ • put in terms of x

Problem 121

$$\int \ln(1-\sqrt{x})dx$$

Start by making the substitution u = $\sqrt{x}$ and du = $\frac{1}{2\sqrt{x}}$ dx

$$2\int u * \ln(1-u)du$$

Make the second substitution w = $1-u$ and dw = -du

$$2\int (w-1)\ln(w)dw$$

Expand out the integral

$$2\int (w * \ln(w) - \ln(w))dw$$

Split the equation into two smaller integrals

$$2\int (w * \ln(w)dw - 2\int \ln(w)\, dw$$

Integrate the integral of w*ln(w) by parts, making f = ln(w), df = $\frac{1}{w}dw$, dg = wdw, and g = $\frac{w^2}{2}$

$$2\frac{w^2}{2}\ln(w) - \int w\, dw - 2\int \ln(w)\, dw$$

Integrate the integral of ln(w) by parts, making f = ln(w), df = $\frac{1}{w}dw$, dg = dw, and g = w

$$2\frac{w^2}{2}\ln(w) - \int w\, dw - 2w\ln(w) + 2\int 1\, dw$$

The integral of w is evaluated to be $\frac{w^2}{2}$, and the integral of 1 is evaluated to be w

$$2\frac{w^2}{2}\ln(w) - \frac{w^2}{2} - 2w\ln(w) + 2w + C$$

Substitute $1-u$ back in for w, and $\sqrt{x}$ back in for u

$$(1-\sqrt{x})^2\ln(1-\sqrt{x}) - \frac{(1-\sqrt{x})^2}{2} - 2(1-\sqrt{x})\ln(1-\sqrt{x}) + 2(1-\sqrt{x}) + C$$

Possible intermediate steps:

$$\int \frac{x^2}{(1+x^2)^3}\, dx$$

For the integrand, $\frac{x^2}{(x^2+1)^3}$, substitute $x = \tan(u)$ and $dx = du \sec^2(u)$. Then $(x^2+1)^3 = (\tan^2(u)+1)^3 = \sec^6(u)$ and $u = \tan^{-1}(x)$:

$$= \int \sin^2(u)\cos^2(u)\, du$$

Write $\cos^2(u)$ as $1 - \sin^2(u)$:

$$= \int \sin^2(u)\left(1 - \sin^2(u)\right) du$$

Expanding the integrand $\sin^2(u)\left(1 - \sin^2(u)\right)$ gives $\sin^2(u) - \sin^4(u)$:

$$= \int \left(\sin^2(u) - \sin^4(u)\right) du$$

Integrate the sum term by term and factor out constants:

$$= \int \sin^2(u)\, du - \int \sin^4(u)\, du$$

Use the reduction formula, $\int \sin^m(u)\, du = -\frac{\cos(u)\sin^{m-1}(u)}{m} + \frac{m-1}{m}\int \sin^{-2+m}(u)\, du$, where $m = 4$:

$$= \frac{1}{4}\sin^3(u)\cos(u) + \frac{1}{4}\int \sin^2(u)\, du$$

Write $\sin^2(u)$ as $\frac{1}{2} - \frac{1}{2}\cos(2u)$:

$$= \frac{1}{4}\sin^3(u)\cos(u) + \int \sin^2(u)\, du - \frac{3}{4}\int\left(\frac{1}{2} - \frac{1}{2}\cos(2u)\right) du$$

Integrate the sum term by term and factor out constants:

$$= \frac{1}{4}\sin^3(u)\cos(u) - \frac{3}{4}\int \frac{1}{2}\, du + \int \sin^2(u)\, du + \frac{3}{8}\int \cos(2u)\, du$$

For the integrand $\cos(2u)$, substitute $s = 2u$ and $ds = 2du$:

$$= \frac{3}{16}\int \cos(s)\, ds + \frac{1}{4}\sin^3(u)\cos(u) - \frac{3}{4}\int \frac{1}{2}\, du + \int \sin^2(u)\, du$$

The integral of $\frac{1}{2}$ is $\frac{u}{2}$:

$$= \frac{3}{16}\int \cos(s)\, ds - \frac{3u}{8} + \frac{1}{4}\sin^3(u)\cos(u) + \int \sin^2(u)\, du$$

The integral of $\cos(s)$ is $\sin(s)$:

$$= \frac{3\sin(s)}{16} - \frac{3u}{8} + \frac{1}{4}\sin^3(u)\cos(u) + \int \sin^2(u)\, du$$

Write $\sin^2(u)$ as $\frac{1}{2} - \frac{1}{2}\cos(2u)$:

$$= \frac{3\sin(s)}{16} - \frac{3u}{8} + \frac{1}{4}\sin^3(u)\cos(u) + \int\left(\frac{1}{2} - \frac{1}{2}\cos(2u)\right) du$$

Integrate the sum term by term and factor out constants:

$$= \frac{3\sin(s)}{16} - \frac{3u}{8} + \frac{1}{4}\sin^3(u)\cos(u) + \int \frac{1}{2}\, du - \frac{1}{2}\int \cos(2u)\, du$$

For the integrand $\cos(2u)$, substitute $p = 2u$ and $dp = 2du$:

$$= -\frac{1}{4}\int \cos(p)\, dp + \frac{3\sin(s)}{16} - \frac{3u}{8} + \frac{1}{4}\sin^3(u)\cos(u) + \int \frac{1}{2}\, du$$

The integral of $\cos(p)$ is $\sin(p)$:

$$= -\frac{\sin(p)}{4} + \frac{3\sin(s)}{16} - \frac{3u}{8} + \int \frac{1}{2}\, du + \frac{1}{4}\sin^3(u)\cos(u)$$

The integral of $\frac{1}{2}$ is $\frac{u}{2}$:

$$= -\frac{\sin(p)}{4} + \frac{3\sin(s)}{16} + \frac{u}{8} + \frac{1}{4}\sin^3(u)\cos(u) + \text{constant}$$

Substitute back for $p = 2u$:

$$= \frac{3\sin(s)}{16} + \frac{u}{8} - \frac{1}{4}\sin(2u) + \frac{1}{4}\sin^3(u)\cos(u) + \text{constant}$$

Substitute back for $s = 2u$:

$$= \frac{u}{8} + \frac{1}{4}\sin^3(u)\cos(u) - \frac{1}{8}\sin(u)\cos(u) + \text{constant}$$

Substitute back for $u = \tan^{-1}(x)$:

$$= \frac{x(x^2-1) + (x^2+1)^2\tan^{-1}(x)}{8(x^2+1)^2} + \text{constant}$$

Which is equal to:

$$= \frac{x}{8(x^2+1)} - \frac{x}{4(x^2+1)^2} + \frac{1}{8}\tan^{-1}(x) + \text{constant}$$

wolfram alpha

123) $\int x \tan^{-1}(x-1)\, dx$

$u = \tan^{-1}(x-1)$ $\quad du = \dfrac{1}{x^2-2x+2}$

$dv = x$ $\quad v = \frac{1}{2}x^2$

$$\frac{1}{2}x^2 \tan^{-1}(x-1) - \int \frac{1}{2}x^2 \frac{1}{x^2-2x+2} =$$

$$\frac{1}{2}x^2 \tan^{-1}(x-1) - \int \frac{x^2}{2(x^2-2x+2)} =$$

$$\frac{1}{2}x^2 \tan^{-1}(x-1) - \frac{\ln|x^2-2x+2| + x}{2}$$

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$$\int \frac{dx}{2x^2-2x+1}$$

$$= \int \frac{1}{\left(\sqrt{2}\,x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$\left\{ \text{substitute } u = \sqrt{2}\,x - \frac{1}{\sqrt{2}} \quad du = \sqrt{2}\,dx \right\}$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+\frac{1}{2}}\,du$$

$$= \tan^{-1}(\sqrt{2}\,u) + c$$

$$\left\{ \text{substitute } u = \sqrt{2}\,x - \frac{1}{\sqrt{2}} \right\}$$

$$\boxed{= -\tan^{-1}(1-2x) + c}$$

Wolfram Alpha.

Page 368, Problem Number One-Hundred and Twenty-Five

Integrate: $\int \frac{-x^2}{\sqrt{1-x^2}} dx$

You should begin this problem by making the u substitution of x= $\sin u$ and $dx = \cos u \, du$ to get:

$$\int \frac{-\sin^2 u \cos u}{\sqrt{1-\sin^2 u}} du = \int -\sin^2 u \, du \quad \text{(By using trigonometric identities)}$$

You should get rid of the square on the sine term by putting it in terms of $\cos 2u$

$$\int \frac{\cos 2u - 1}{2} du$$

Divide the integral into two integrals to simplify integration, and integrate them to get:

$$\frac{\sin 2u}{4} - \frac{1}{2} u + C$$

Using the fact that $\sin 2u = 2 \sin u \cos u$ you can re-write this as:

$$\frac{\sin u \cos u}{2} - \frac{1}{2} u + C$$

At this point you put u back into terms of x to get:

$$\frac{\sin(\sin^{-1}(x)) \cdot \cos(\sin^{-1}(x))}{2} - \frac{1}{2} \sin^{-1} x + C$$

At this point, simplify your equation. Remember that $\cos u = \sqrt{1 - \sin^2 u}$ and that $\sin^2 u = x^2$ from our original u substitution. This should allow you to get the final answer of:

$$\boldsymbol{\frac{1}{2}\left(x\sqrt{1 - x^2} - \sin^{-1}(x)\right) + C}$$