Spivak page 381 #1 (vi).

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+\left(\frac{x}{a}\right)^2}$$

Let $u=\frac{x}{a}$; $x=au$; $dx=a\,du$.

$$= \frac{1}{a^2}\int \frac{a\,du}{1+u^2} = \frac{1}{a}\int \frac{du}{1+u^2}$$

$$= \frac{1}{a}\arctan u$$

$$= \frac{1}{a}\arctan \frac{x}{a}$$

Spivak page 381 #2 (vi).

$$\int \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

Let $u=x^2$ then $du = 2x\,dx$

Let $u = \sin v$ then $du = \cos v\, dv$

$$= \frac{1}{2}\int \frac{\cos v\, dv}{\sqrt{1-\sin^2 v}} = \frac{1}{2}\int \frac{\cos v\, dv}{\sqrt{\cos^2 v}}$$

$$= \frac{1}{2}\int dv = \frac{1}{2} v = \frac{1}{2}\arcsin u$$

$$= \frac{1}{2}\arcsin x^2$$

Spivak page 381 # 3(vi)

$$\int \frac{\log(\log x)}{x}\,dx$$

Let $u = \log(\log x)$ $\qquad du = \frac{1}{\log x}\frac{1}{x}\,dx$

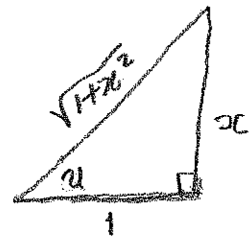
$dv = \frac{1}{x}\,dx$ $\qquad v = \log x$

$$= (\log x)(\log(\log x)) - \int \frac{1}{x}\,dx$$

$$= (\log x)(\log(\log x)) - \log x$$

Spivak page 382 #4 (vi)

$$\int \frac{dx}{x\sqrt{1+x^2}}$$



$\tan u = x$

$\cot u = \frac{1}{x}$

$\csc u = \frac{\sqrt{1+x^2}}{x}$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

Let $x = \tan u$ then $dx = \sec^2 u \, du$

$$= \int \frac{\sec^2 u \, du}{\tan u \sqrt{1+\tan^2 u}} = \int \frac{\sec^2 u}{\tan u \sqrt{\sec^2 u}} \, du$$

$$= \int \frac{1}{\cos u} \cdot \frac{\cos u}{\sin u} \, du = \int \csc u \, du$$

$$= -\log(\csc u + \cot u)$$

$$= -\log\left(\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right)$$

$$= \log x - \log(1 + \sqrt{1+x^2})$$

Spivak page 383 #5(vi)

$$\int \frac{dx}{\sqrt{\sqrt{x}+1}}$$

Let $u = \sqrt{\sqrt{x}+1}$; $u^2 = \sqrt{x}+1$; $u^2-1 = \sqrt{x}$

$x = (u^2-1)^2$; $dx = 2(u^2-1)2u\,du$

$$= \int \frac{4\cancel{u}(u^2-1)\,du}{\cancel{u}} = 4\int (u^2-1)\,du$$

$$= 4\left(\frac{1}{3}u^3 - u\right)$$

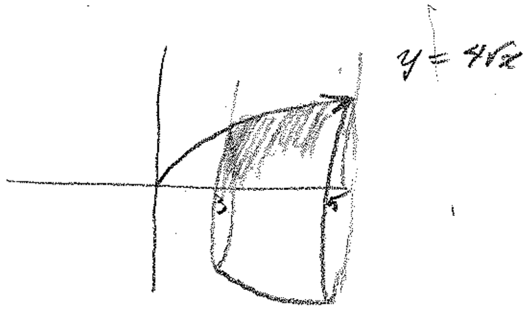
$$= 4\left(\frac{1}{3}\left(\sqrt{\sqrt{x}+1}\right)^3 - \sqrt{\sqrt{x}+1}\right)$$

$$= 4\sqrt{\sqrt{x}+1}\left(\frac{1}{3}(\sqrt{x}+1) - 1\right)$$

$$= 4\sqrt{\sqrt{x}+1}\left(\frac{1}{3}\sqrt{x} - \frac{2}{3}\right)$$

$$= \frac{4}{3}\left(\sqrt{\sqrt{x}+1}\right)(\sqrt{x} - 2)$$

Klein page 441 #2. The upper half of the parabola $y^2 = 16x$ is rotated around the x-axis. Find the volume of the solid generated by the arc lying between $x=3$ and $x=5$.

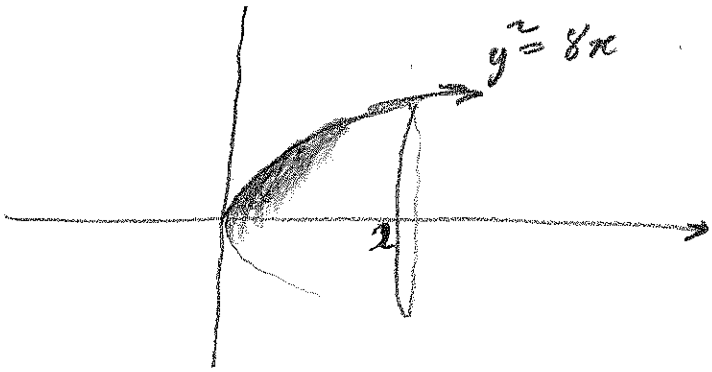


$$V = \int_3^5 \pi y^2 \, dx = \int_3^5 16\pi x \, dx = 8\pi x^2 \Big|_3^5$$

$$= 8\pi (25 - 9) = 112\pi$$

$$\begin{array}{r} {}^{3}14 \\ \times\ 8 \\ \hline 112 \end{array}$$

Klein page 441 # 4. Find the volume generated by rotating the area bounded by $y^2 = 8x$, $x = 2$ and the x-axis about the y-axis.

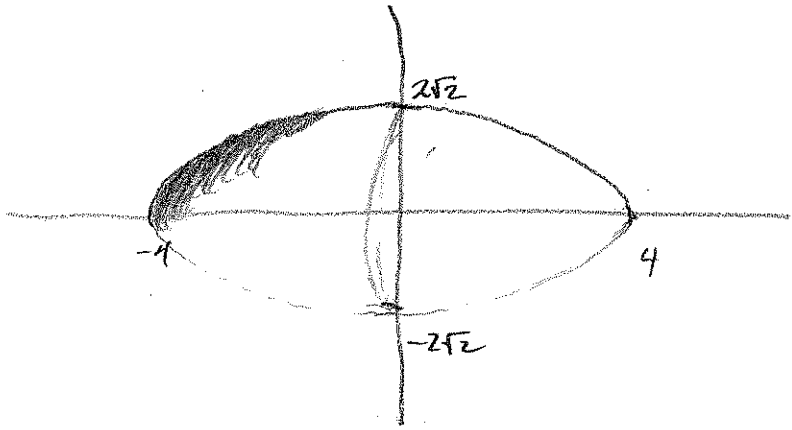


$$V = \int_0^2 \pi y^2 dx = \int_0^2 8\pi x dx = 4\pi x^2 \Big|_0^2 = 16\pi$$

Klein page 441 #6. The upper half of the ellipse $3x^2+6y^2=48$ is rotated about the x-axis. The surface it generates is another kind of ellipsoid called an oblate spheriod. Find the volume of the spheroid.

If $x=0$ then $y^2=8$

$y=\pm 2\sqrt{2}$



If $y=0$ then $x^2=16$

$x=\pm 4$

$$V=\int_{-4}^{4}\pi y^2\,dx=\int_{-4}^{4}\pi\left(\frac{48-3x^2}{6}\right)dx=\int_{0}^{4}\pi\left(\frac{48-3x^2}{3}\right)dx$$

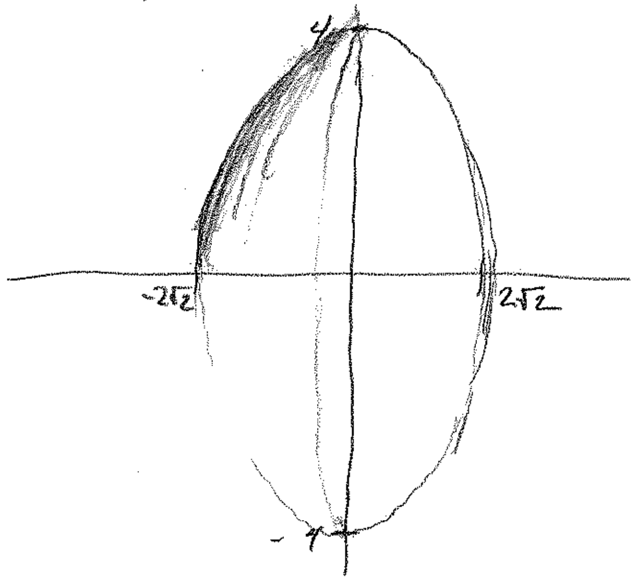
where the last equality above is due to symmetry.

$$\frac{\pi}{3}\int_0^4(48-3x^2)\,dx=\frac{\pi}{3}(48x-x^3)\Big|_0^4$$

$$=\frac{\pi}{3}(48\cdot 4-4^3)=\frac{\pi}{3}(12\cdot 4^2-4^3)$$

$$=\frac{\pi}{3}(3-1)4^3=\frac{2\cdot 4^3\pi}{3}=\frac{128\pi}{3}.$$

Klein page 441 #7. If the upper half of the ellipse $6x^2+3y^2=48$ is rotated around the x-axis, the surface it generates is another kind of ellipsoid called an oblate spheroid. Find the volume of the spheroid.



If $x=0$ then $y^2=16$

$y=\pm 4$

If $y=0$ then $x^2=8$

$x=\pm 2\sqrt{2}$

$$V=\int_{-2\sqrt{2}}^{2\sqrt{2}} \pi y^2 dx = \int_{-2\sqrt{2}}^{2\sqrt{2}} \pi \frac{48-6x^2}{3} dx = \int_0^{2\sqrt{2}} 2\pi \frac{48-6x^2}{3} dx$$

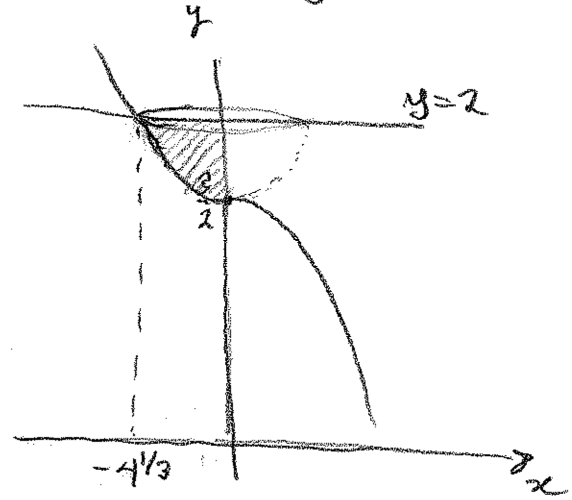
where again the last equality is due to symmetry.

$$\frac{2\pi}{3}\int_0^{2\sqrt{2}} (48-6x^2)dx = \frac{2\pi}{3}(48x-2x^3)\Big|_0^{2\sqrt{2}}$$

$$= \frac{2\pi}{3}\left(48\cdot 2\sqrt{2} - 2\cdot 8\cdot 2\sqrt{2}\right) = \frac{2\pi}{3}\cdot 32(3-1)\sqrt{2}$$

$$= \frac{4\cdot 32}{3}\pi\sqrt{2} = \frac{128\sqrt{2}}{3}\pi .$$

Klein page 447 #2. Consider the area lying between an arc of $y = \frac{12-x^3}{8}$, the y-axis and the line $y=2$. What is the volume of the solid of revolution generated by revolving this area around the y-axis.



y-intercept $y = \frac{12}{8} = \frac{6}{4} = \frac{3}{2}$

Intersection of $y=2$ with arc.

$2 = \frac{12-x^3}{8}$; $16 = 12 - x^3$

$x^3 = 12 - 16 = -4$

$x = -4^{1/3}$

Use shell method

height $= 2 - \frac{12-x^3}{8} = \frac{4+x^3}{8}$

$$V = \int_{-4^{1/3}}^{0} 2\pi(-x)\left(\frac{4+x^3}{8}\right)dx = \frac{\pi}{4}\int_{0}^{-4^{1/3}} (4x + x^4)\,dx$$

$$= \frac{\pi}{4}\left(2x^2 + \frac{x^5}{5}\right)\Bigg|_{0}^{-4^{1/3}} = \frac{\pi}{4}\left(2\cdot 4^{2/3} - 4\cdot\frac{4^{2/3}}{5}\right)$$

$$= \frac{\pi}{10}\left(5\cdot 4^{2/3} - 2\cdot 4^{2/3}\right) = \frac{3\pi}{10}\,4^{2/3}.$$