Math 182 Honors Quiz 13 Version A

1. State Taylor's Theorem with the integral form of the remainder term.

Suppose f has $n+1$ continuous derivatives. Then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

2. State the ratio test for determining whether an infinite series converges.

If $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n < \infty$.

3. Prove the integration by parts formula: If f' and g' are continuous then

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

By the chain rule $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$.

Since f' and g' are continuous, then so is $\frac{d}{dx}(f(x)g(x))$.

It follows, from the fundamental theorem of Calculus

$$f(x)g(x) = \int \frac{d}{dx} f(x)g(x)\,dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$

Therefore

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

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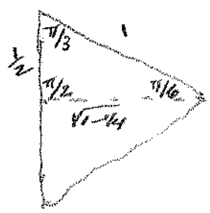
4. Solve the following integration problems:

(i) $\int_0^4 \sqrt{1+2x}\,dx = \frac{1}{2}\int_1^9 \sqrt{u}\,du = \frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{2}\frac{2}{3}u^{3/2}\Big|_1^9$

$u = 1+2x$; $du = 2dx$

$$= \frac{1}{3}u^{3/2}\Big|_1^9 = \frac{1}{3}9^{3/2} - \frac{1}{3} = \frac{1}{3}3^3 - \frac{1}{3} = 9 - \frac{1}{3} = \frac{26}{3}$$

(ii) $\int_0^{\pi/6} \sin^2 x\,dx = \int_0^{\pi/6} \frac{1-\cos 2x}{2}\,dx = \frac{\pi}{12} - \frac{1}{2}\int_0^{\pi/6}\cos 2x\,dx$



$$= \frac{\pi}{12} - \frac{1}{4}\sin 2x\Big|_0^{\pi/6} = \frac{\pi}{6} - \frac{1}{4}\sin \pi/3 = \frac{\pi}{6} - \frac{1}{4}\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

(iii) $\int x\arctan(1+x^2)\,dx = \frac{1}{2}\int \arctan u\,du$

$u = 1+x^2$; $du = 2x\,dx$

$p = \arctan u$; $dp = \frac{1}{1+u^2}du$

$dq = du$; $q = u$

$$= \frac{1}{2}\left(u\arctan u - \int \frac{u}{1+u^2}du\right) = \frac{1}{2}u\arctan u - \frac{1}{4}\ln(1+u^2)$$

$$= \frac{1}{2}(1+x^2)\arctan(1+x^2) - \frac{1}{4}\ln(1+(1+x^2)^2)$$

(iv) $\int \frac{x^3}{x^2-1}\,dx = \int\left(x + \frac{x}{x^2-1}\right)dx = \frac{x^2}{2} + \int \frac{x}{x^2-1}\,dx$

$x + \frac{x}{x^2-1}$

$x^2-1 \overline{)\,x^3}$

$x^3 - x$

x

$$= \frac{x^2}{2} + \frac{1}{2}\int \frac{2x}{x^2-1}\,dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1|$$

5. Find the Taylor series with remainder for $f(x) = \ln(1+x^2)$ expanded about $a = 0$.

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n + \frac{t^{n+1}}{1-t}$$

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^{n+1} t^{n-1} + (-1)^n \frac{t^n}{1+t}$$

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{t^n}{n}$$

$$+ \int_0^x (-1)^n \frac{t^n}{1+t}\, dt.$$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots + (-1)^{n-1}\frac{t^{2n}}{n} + \int_0^{x^2} (-1)^n \frac{t^n}{1+t}\, dt$$

6. Find $\lim_{x \to 0} \dfrac{xe^{-x^2} - \sin x}{x^3}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$xe^{-x^2} = \cancel{x} - x^3 + \frac{x^5}{2!} - \cdots$$

$$\sin x = \cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{xe^{-x^2} - \sin x}{x^3} = \frac{\left(-x^3 + \frac{x^5}{2!} - \cdots\right) - \left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^3}$$

$$= \left(-1 + \frac{1}{6}\right) + \left(\frac{1}{2} - \frac{1}{5!}\right)x + \cdots$$

$$\to -\frac{5}{6} \text{ as } x \to 0.$$

7. Determine whether the following infinite series converge and explain your answer.

(i) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by the integral test

Since $\int_1^\infty \frac{1}{x^3} dx = \int_1^\infty x^{-3} dx = \frac{1}{-2} x^{-2} \Big|_1^\infty = \frac{1}{2} < \infty$

Then $\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty.$

(ii) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+1)^{3/2}}$ by the limit comparison test

Since $\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{(n+1)^{3/2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{3/2}}{(n+1)^{3/2}} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^{3/2} = 1$

Then $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ implies $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+1)^{3/2}} = \infty$.

(iii) $\sum_{n=237}^{\infty} \frac{n^2-1}{n!}$ be the ratio test

Since $\lim_{n\to\infty} \frac{(n+1)^2-1}{(n+1)!} \Big/ \frac{n^2-1}{n!} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{(n+1)^2-1}{n^2-1} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{(1+\frac{1}{n})^2 - \frac{1}{n^2}}{1-\frac{1}{n^2}}$

$= 0 \cdot 1 = 0 < 1$

Then $\sum_{n=237}^{\infty} \frac{n^2-1}{n!} < \infty$

(iv) $\sum_{n=13}^{\infty} \frac{1}{(\ln n)^n}$ by the root test

Since $\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$

then $\sum_{n=13}^{\infty} \frac{1}{(\ln n)^n} < \infty$

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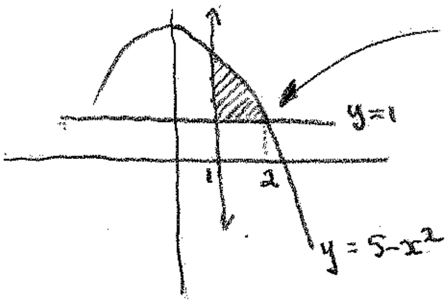
8. Find the volume generated by rotating the region in the first quadrant bounded by $x = 1$, $y = 1$ and $y = 5 - x^2$ about the x-axis.



Point of intersection

$$\begin{cases} y = 5 - x^2 \\ y = 1 \end{cases}$$

$$5 - x^2 = 1$$

$$x^2 = 4$$

$$x = 2$$

$$V = \int_1^2 \pi(y^2 - 1^2)\,dx = \int_1^2 \pi((5-x)^2 - 1)\,dx = \int_1^2 \pi(x^4 - 10x^2 + 25)\,dx - \pi$$

$$= \pi\left(\frac{x^5}{5} - \frac{10}{3}x^3 + 25x\right)\Big|_1^2 - \pi = \pi\left(\frac{2^5}{5} - \frac{5\cdot 2^4}{3} + 50 - \frac{1}{5} + \frac{5\cdot 2}{3} - 25\right) - \pi$$

$$= \pi\left(\frac{31}{5} - \frac{5}{3}(14) + 25\right) - \pi = \pi\left(\frac{93 - 25(14) + 25\cdot 15}{15}\right) - \pi = \frac{118}{15}\pi - \pi = \frac{103}{15}\pi$$

9. Find the length of the arc given be $y = \frac{1}{8}x^2 - \ln x$ between $x = 1$ and $x = 2$.

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1 + \left(\frac{1}{4}x - \frac{1}{x}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{1}{4}x\right)^2 - \frac{1}{2} + \frac{1}{x^2}}\,dx = \int_1^2 \sqrt{\left(\frac{1}{4}x\right)^2 + \frac{1}{2} + \frac{1}{x^2}}\,dx$$

$$= \int_1^2 \sqrt{\left(\frac{1}{4}x + \frac{1}{x}\right)^2}\,dx = \int_1^2 \left(\frac{1}{4}x + \frac{1}{x}\right)dx$$

$$= \frac{1}{8}x^2 + \ln x \Big|_1^2 = \frac{1}{8}(4) + \ln 2 - \frac{1}{8} = \frac{3}{8} + \ln 2.$$