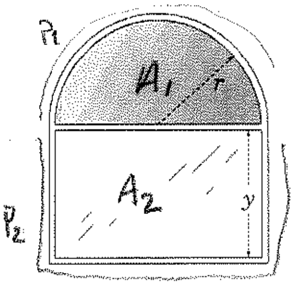
Math 182 Quiz 2 Version B — Key

1. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, wheras the semicircle is of tinted glass that transmits only half as much light per unit area as the clear glass does. The total perimeter is fixed.

$P_1 = \pi r$

$P_2 = 2r + 2y$



$A_1 = \frac{\pi}{2} r^2$

$A_2 = 2yr$

For the variables r and y indicated above find the proportion of r to y for the window that will admit the most light. Neglect the thickness of the frame.

Light: $L = \frac{1}{2}A_1 + A_2 = \frac{\pi}{4} r^2 + 2yr$

Perimeter: $P = P_1 + P_2 = \pi r + 2r + 2y$

$$2y = P - (2+\pi)r$$

Thus, $L = \frac{\pi}{4} r^2 + (P - (2+\pi)r)r$

$$\frac{dL}{dr} = \frac{\pi}{2} r + P - 2(2+\pi)r = 0$$

$$\left(4 + 2\pi - \frac{\pi}{2}\right) r = P$$

$$r = \frac{P}{4 + \frac{3\pi}{2}} = \frac{2\cancel{P}}{8 + 3\pi}$$

$$y = \frac{1}{2}\left(P - (2+\pi)\frac{2P}{8+3\pi}\right) = \frac{P}{2}\left(\frac{8+3\pi-4-2\pi}{8+3\pi}\right) = \frac{\cancel{P}}{2}\left(\frac{4+\pi}{8+3\pi}\right)$$

The proportion of r to y is

$$\frac{r}{y} = \frac{\left(\frac{2\cancel{P}}{\cancel{8+3\pi}}\right)}{\frac{\cancel{P}}{2}\left(\frac{4+\pi}{\cancel{8+3\pi}}\right)} = \frac{4}{4+\pi}$$

in otherwords the proportion is 4 to $4+\pi$.

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2. State the Mean Value Theorem (for derivatives).

If f is continuous on $[a,b]$ and differentiable on (a,b) then there is at least one point c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

3. State the definition of $\ln x$ in terms of an integral.

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

4. Use the above definition of logarithm to show that $\ln a + \ln b = \ln ab$.

$$\ln b = \int_1^b \frac{1}{t}\,dt$$

substitute $u = at$ so $du = a\,dt$

$$'' = \int_a^{ab} \frac{1}{\left(\frac{u}{a}\right)} \cdot \frac{1}{a}\,du = \int_a^{ab} \frac{1}{u}\,du.$$

Therefore

$$\ln a + \ln b = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{u}\,du$$

$$= \int_1^{ab} \frac{1}{t}\,dt = \ln(ab).$$

5. Solve the following indefinite integrals:

(i) $\displaystyle\int \sqrt{\frac{x^2}{1+x^2}}\,dx = \int \frac{|x|}{\sqrt{1+x^2}}\,dx = \begin{cases} \int \frac{x}{\sqrt{1+x^2}}\,dx & \text{if } x \geq 0 \\ -\int \frac{x}{\sqrt{1+x^2}}\,dx & \text{if } x \leq 0 \end{cases}$

Now let $u = 1+x^2$ so $du = 2x\,dx$. Since

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + C = \sqrt{1+x^2} + C$$

then

$$\int \sqrt{\frac{x^2}{1+x^2}}\,dx = \begin{cases} \sqrt{1+x^2} + C_1 & \text{if } x \geq 0 \\ -\sqrt{1+x^2} + C_2 & \text{if } x \leq 0 \end{cases} = \begin{cases} \sqrt{1+x^2} & \text{if } x \geq 0 \\ -\sqrt{1+x^2} + 2 & \text{if } x \leq 0 \end{cases} + C.$$

Solve for C_1 and C_2 so the pieces match at $x=0$.

(ii) $\displaystyle\int \frac{\ln x^3}{x}\,dx$

$$= 3\int \frac{\ln x}{x}\,dx = 3\int u\,du = \frac{3}{2}u^2 + C = \frac{3}{2}(\ln x)^2 + C.$$

Let $u = \ln x$ $\quad du = \frac{1}{x}dx$

(iii) $\displaystyle\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx$

Let $u = \sin x$ $\quad du = \cos x\,dx$

$$= \int (1-u^2)\,du = u + \frac{u^3}{3} + C = \sin x + \frac{\sin^3 x}{3} + C$$

(iv) $\displaystyle\int \ln(2-x)\,dx = -\int \ln u\,du = -(u\ln u - u) + C$

Let $u = 2-x$ so $du = -dx$

$$= -u\ln u + u + C = -(2-x)\ln(2-x) + 2 - x + C$$

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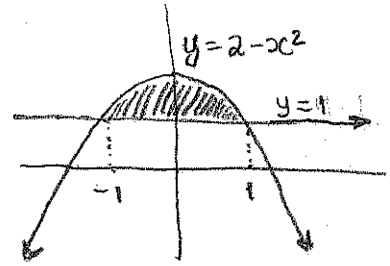
6. Find the volume of the solid generated by revolving the region bounded by $y = 2 - x^2$ and $y = 1$ about the x-axis.

With disk/washer method:



$$V = \int_{-1}^{1} \pi\left((2-x^2)^2 - 1\right)dx = 2\int_0^1 \pi\left((2-x^2)^2-1\right)dx$$

$$= 2\pi\int_0^1 (x^4 - 4x^2 + 3)\,dx = 2\pi\left(\frac{x^5}{5} - \frac{4}{3}x^3 + 3x\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{5} - \frac{4}{3} + 3\right) = 2\pi\left(\frac{3-20+45}{15}\right) = \frac{56\pi}{15}$$

$2 - x^2 = 1$

$x = \pm 1$

$x^2 = 2 - y$

$x = \pm\sqrt{2-y}$

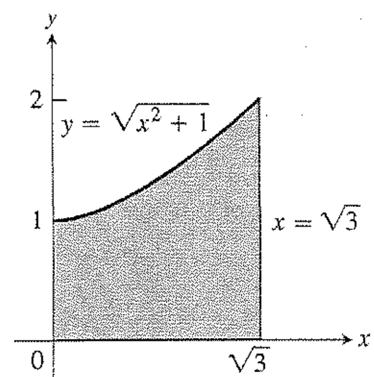
With shell method:

$$V = \int_1^2 2\pi y\left(2\sqrt{2-y}\right)dy = -4\pi\int_1^0 (2-u)\sqrt{u}\,du = 4\pi\int_0^1 (2u^{1/2} - u^{3/2})\,du$$

$u = 2-y$, $du = -dy$

$$= 4\pi\left(\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^1 = 4\pi\left(\frac{4}{3} - \frac{2}{5}\right) = \frac{56\pi}{15}$$

7. Find the volume of the solid generated by revolving the the shaded region about the y-axis.



shell method:

$$V = \int_0^{\sqrt{3}} 2\pi x\sqrt{x^2+1}\,dx$$

$u = x^2 + 1 \qquad du = 2x\,dx$

$$= \pi\int_1^4 \sqrt{u}\,du = \frac{2\pi}{3}u^{3/2}\Big|_1^4$$

$$= \frac{2\pi}{3}(2^3 - 1) = \frac{14\pi}{3}.$$