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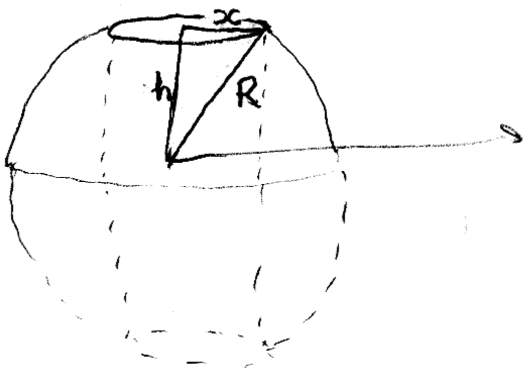
Math 182 Quiz 3 Version B ~ Key ~

1. A cylindrical hole of radius x is bored through a sphere of radius R in such a way that the axis of the hole passes through the center of the sphere. Find the value of x that maximizes the complete surface area of the remaining solid.



Area of top half of sphere...

$$A_1 = \int_0^h 2\pi\sqrt{R^2-y^2}\sqrt{\frac{y^2}{R^2-y^2}+1}\,dy$$

$$= \int_0^h 2\pi\sqrt{\cancel{y^2}+R^2-\cancel{y^2}}\,dy$$

$$= \int_0^h 2\pi R\,dy = 2\pi R h$$

$$\frac{d\sqrt{R^2-y^2}}{dy} = \frac{-y}{\sqrt{R^2-y^2}}$$

$$h = \sqrt{R^2-x^2}$$

Area of top half of cylinder...

$$A_2 = 2\pi x h.$$

Total area is:

$$A = 2A_1 + 2A_2 = 4\pi R h + 4\pi x h = 4\pi(R+x)\sqrt{R^2-x^2}$$

$$\frac{dA}{dx} = 4\pi\sqrt{R^2-x^2} + 4\pi(R+x)\frac{-x}{\sqrt{R^2-x^2}} = 0$$

$$R^2 - x^2 = x(R+x) = xR + x^2$$

$$2x^2 + xR - R^2 = 0$$

$a = 2 \quad b = R \quad c = -R^2$

$$x = \frac{-R \pm \sqrt{R^2+8R^2}}{4} = \frac{-R+3R}{4} = \frac{2R}{4} = \frac{R}{2}.$$

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2. State the Fundamental Theorem of Calculus Part II.

Suppose f is continuous on $[a,b]$ and F is any antiderivative of F on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

3. State the definition of $\sinh t$ in terms of exponential functions.

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

4. Use the quadratic formula and logarithms to solve for $t = \sinh^{-1}(s)$.

$$s = \frac{e^t - e^{-t}}{2}, \qquad 2s = e^t - e^{-t}, \qquad w = e^t$$

Thus $2s = w - \frac{1}{w}$ or $w^2 - 2sw - 1 = 0$

$a = 1 \quad b = -2s \quad c = -1$

Therefore $w = \dfrac{2s \pm \sqrt{4s^2+4}}{2} = s \pm \sqrt{s^2+1}$

Choose $\oplus$ solution since $e^t > 0$.

$$e^t = s + \sqrt{s^2+1}$$

$$t = \ln(s + \sqrt{s^2+1})$$

$$\sinh^{-1}(s) = \ln(s + \sqrt{s^2+1}).$$

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5. Solve the following indefinite integrals:

(i) $\int x^2\sqrt{x+1}\,dx \quad = \int (u-1)^2 \sqrt{u}\, du$

$u = x+1 \qquad x = u-1$

$du = dx$

$= \int (u^2 - 2u + 1) u^{1/2}\, du = \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$

$= \frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C$

$= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C$

(ii) $\int xe^{2x}\,dx$

by parts

$u = x \qquad du = dx$

$dv = e^{2x}dx \quad v = \frac{1}{2}e^{2x}$

$= \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx$

$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C.$

(iii) $\int \frac{1}{x^2+4}\,dx \quad = \frac{1}{4}\int \frac{1}{1+(\frac{x}{2})^2}\,dx$

$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

(iv) $\int \sqrt{4-x^2}\,dx \quad = 2\int \sqrt{1-(\frac{x}{2})^2}\,dx = 4\int \sqrt{1-\sin^2 u}\cos u\, du$

$\sin u = \frac{x}{2} \qquad \cos u\, du = \frac{dx}{2}$

$= 4\int \frac{1+\cos 2u}{2}\,du = 2u + \sin 2u = 2u + 2\sin u\cos u$

$= 2u + 2\sin u\sqrt{1-\cos^2 u} \quad = 2\arcsin\frac{x}{2} + \underbrace{2\cdot\frac{x}{2}\sqrt{1-(\frac{x}{2})^2}}$

$= 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$

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6. Find the length of the curve given by $y = \frac{1}{12}x^3 + x^{-1}$ between $x = 1$ and $x = 2$.

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1+\left(\frac{1}{4}x^2 - \frac{1}{x^2}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4}}\,dx = \int_1^2 \sqrt{\frac{1}{16}x^4 + \frac{1}{2} + \frac{1}{x^4}}\,dx$$

$$= \int_1^2 \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2}\,dx = \int_1^2 \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{12}x^3 - \frac{1}{x}\Big|_1^2 = \frac{1}{12}8 - \frac{1}{2} - \frac{1}{12} + 1$$

$$= \frac{8-6-1+12}{12} = \frac{13}{12}.$$

$$y = \sqrt{x} - \frac{1}{3}x^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2}x^{1/2}$$

7. Find the area of the surface of revolution generated by revolving the curve $y = \frac{1}{3}\sqrt{x}(3-x)$ between $x = 0$ and $x = 3$ about the x-axis.

$$A = \int_0^3 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \frac{2\pi}{3}\sqrt{x}(3-x)\sqrt{1+\left(\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^2}\,dx$$

$$= \int_0^3 2\pi\left(\sqrt{x} - \frac{1}{3}x^{3/2}\right)\sqrt{1+\frac{1}{4x} - \frac{1}{2} + \frac{x}{4}}\,dx$$

$$= \int_0^3 2\pi\left(\sqrt{x} - \frac{1}{3}x^{3/2}\right)\sqrt{\frac{1}{4x} + \frac{1}{2} + \frac{x}{4}}\,dx = 2\pi\int_0^3 \left(\sqrt{x} - \frac{1}{3}x^{3/2}\right)\left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right)dx$$

$$= 2\pi\int_0^3 \left(\frac{1}{2} + \frac{x}{2} - \frac{1}{6}x - \frac{1}{6}x^2\right)dx = 2\pi\left(\frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{18}x^3\right)\Big|_0^3$$

$$= 2\pi\left(\frac{3}{2} + \frac{9}{3} - \frac{27}{18}\right) = 2\pi\,\frac{54+54-\cancel{54}}{36} = \frac{54\pi}{18} = 3\pi.$$

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$$\begin{array}{r} \overset{2}{18} \\ \times\ 3 \\ \hline 54 \end{array} \qquad \begin{array}{r} \overset{1}{27} \\ 27 \\ \hline 54 \end{array}$$

Alternative solutions to integral problems.

5(i). $\int x^2\sqrt{x+1}\,dx = \frac{2}{3}x^2(x+1)^{3/2} - \int \frac{4}{3}x(x+1)^{3/2}dx$

$u = x^2 \qquad du = 2x\,dx$

$dv = \sqrt{x+1}\,dx \qquad v = \frac{2}{3}(x+1)^{3/2}$

$\int x(x+1)^{3/2}dx = \frac{2}{5}x(x+1)^{5/2} - \int \frac{2}{5}(x+1)^{5/2}dx$

$u = x \qquad du = dx$

$dv = (x+1)^{3/2}dx \qquad v = \frac{2}{5}(x+1)^{5/2}$

$= \frac{2}{5}x(x+1)^{5/2} - \frac{2}{5}\cdot\frac{2}{7}(x+1)^{7/2} + c$

Therefore

$\int x^2\sqrt{x+1}\,dx = \frac{2}{3}x^2(x+1)^{3/2} - \frac{8}{15}x(x+1)^{5/2} + \frac{16}{105}(x+1)^{7/2} + c$

5(ii) $\int xe^{2x}dx = (ax+b)e^{2x} + c$

$\frac{d}{dx}(ax+b)e^{2x} = ae^{2x} + (ax+b)2e^{2x}$

$= (2ax + 2b + a)e^{2x} = xe^{2x}$

$2a = 1 \qquad a = 1/2$

$2b + a = 0 \qquad 2b = -\frac{1}{2} \qquad b = -1/4$

Therefore

$\int xe^{2x}dx = (\frac{1}{2}x - 1/4)e^{2x} + c$

Alternative solutions continued...

5(iii) $\int \frac{1}{x^2+4}\,dx = \int \frac{2\sec^2 u}{(2\tan u)^2+4}\,du$

$x = 2\tan u \qquad u = \arctan\frac{x}{2}$

$dx = 2\sec^2 u\,du$

$= \int \frac{2\sec^2 u}{4(\tan^2 u+1)}\,du = \frac{1}{2}\int \frac{\cancel{\sec^2 u}}{\cancel{\sec^2 u}}\,du$

since $\tan^2 u + 1 = \sec^2 u$

$= \frac{1}{2}\int du = \frac{1}{2}u + C = \frac{1}{2}\arctan\frac{x}{2} + C.$

Alternative solutions continues...

5(iv) $\int \sqrt{4-x^2}\,dx = x\sqrt{4-x^2} + \int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

$u = \sqrt{4-x^2}$ $\qquad du = \frac{-x}{\sqrt{4-x^2}}\,dx$

$dv = dx$ $\qquad v = x$

Therefore

$$\int \sqrt{4-x^2}\,dx - \int \frac{x^2}{\sqrt{4-x^2}}\,dx = x\sqrt{4-x^2}$$

On the other hand

$$\int \sqrt{4-x^2}\,dx + \int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4-x^2+x^2}{\sqrt{4-x^2}}\,dx$$

$$= \int \frac{4}{\sqrt{4-x^2}}\,dx = 2\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = 4\arcsin\frac{x}{2}$$

Adding these equations

$$2\int \sqrt{4-x^2}\,dx = x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}$$

Therefore

$$\int \sqrt{4-x^2}\,dx = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C.$$