Math 182 Quiz 4 Version B

1. Find the following indefinite integrals:

(i) $\int x^2\sqrt{x+1}\,dx = \int (u-1)^2 \sqrt{u}\,du$

$u = x+1 \qquad x = u-1$

$du = dx$

$\int (u^2 - 2u + 1)\sqrt{u}\,du = \int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$

$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c$

$= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + c$

(ii) $\int 2x\cos(x^2-1)\,dx = \int \cos u\,du = \sin u + c$

$u = x^2 - 1$

$du = 2x\,dx$

$= \sin(x^2-1) + c$

(iii) $\int \frac{1}{x^2+9}\,dx = \frac{1}{9}\int \frac{1}{1+(\frac{x}{3})^2}\,dx$

$u = \frac{x}{3}, \quad du = \frac{dx}{3}, \quad 3du = dx$

$= \frac{1}{9}\int \frac{1}{1+u^2}\,3du = \frac{1}{3}\arctan u + c$

$= \frac{1}{3}\arctan\frac{x}{3} + c$

2. Find the following improper integrals:

(i) $\int_0^1 \frac{x^2-5}{\sqrt{x}}\,dx = \int_0^1 (x^{3/2} - 5x^{-1/2})\,dx$

$= \lim_{r\to 0} \int_r^1 (x^{3/2} - 5x^{-1/2})\,dx$

$= \lim_{r\to 0} \left(\frac{2}{5}x^{5/2} - 10x^{1/2}\right)\Big|_r^1$

$= \lim_{r\to 0} \left(\frac{2}{5} - 10 - \frac{2}{5}r^{5/2} + 10r^{1/2}\right) = \frac{2}{5} - 10 = -\frac{48}{5}$

(ii) $\int_1^\infty \frac{\ln x}{x^2}\,dx = \int_0^\infty \frac{u}{e^u}\,du = \int_0^\infty u\,e^{-u}\,du$

$u = \ln x, \quad du = \frac{1}{x}dx, \quad x = e^u$

Now use integration by parts

$p = u \qquad dp = du$

$dq = e^{-u}du \qquad q = -e^{-u}$

$= -ue^{-u}\Big|_0^\infty + \int_0^\infty e^{-u}\,du = -0 + 0 + -e^{-u}\Big|_0^\infty = 1$

by L'Hôpital $\lim_{r\to\infty} re^{-r} = \lim_{r\to\infty} \frac{r}{e^r} = \lim_{r\to\infty} \frac{1}{e^r} = 0$

(iii) $\int_1^\infty \frac{1}{x(x+5)}\,dx = \int_1^\infty \left(\frac{a}{x+5} + \frac{b}{x}\right)dx$

$\frac{a}{x+5} + \frac{b}{x} = \frac{ax + b(x+5)}{x(x+5)} = \frac{1}{x(x+5)}$

$(a+b)x + 5b = 1 \qquad b = \frac{1}{5},\ a = -\frac{1}{5}$

$= \left(-\frac{1}{5}\ln|x+5| + \frac{1}{5}\ln|x|\right)\Big|_1^\infty = \frac{1}{5}\ln\left(\frac{x}{x+5}\right)\Big|_1^\infty$

$= \frac{1}{5}\ln\left(\frac{1}{1+5/x}\right)\Big|_1^\infty = 0 - \frac{1}{5}\ln\frac{1}{6} = \frac{1}{5}\ln 6$

3. Consider the following theorem from your book:

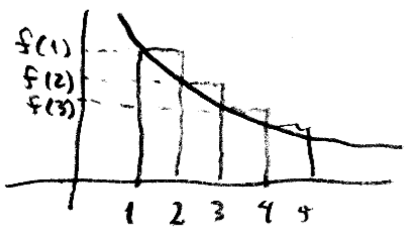
> Theorem 9. Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^\infty f(x)\,dx$ both converge or both diverge.

(i) What is the name of this theorem?

Integral comparison Test

(ii) Establish this theorem for the case $N = 1$.

Since f is continuous, positive and decreasing for $x \geq 1$ then its graph looks like

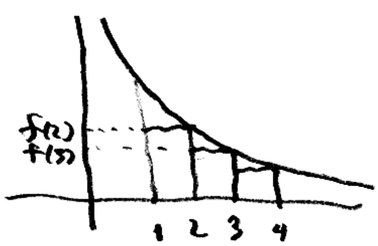


The sum of the areas of the rectangles is $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} a_n$ and this is greater than the area under the curve

Therefore

$$\int_1^\infty f(x)\,dx \leq \sum_{n=1}^{\infty} a_n .$$

If we draw rectangles whose right end point agrees with the curve then



The sum of the areas of the rectangles is $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} a_n$ and this is less than the area under the curve

Therefore

$$\sum_{n=2}^{\infty} a_n \leq \int_1^\infty f(x)\,dx .$$

Combining these results gives

$$\int_1^\infty f(x)\,dx \leq \sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n \leq a_1 + \int_1^\infty f(x)\,dx$$

Therefore the series and the integral either both diverge or they both converge.

Math 182 Quiz 4 Version B

4. Determine whether the following series converge or diverge and explain your answer.

(i) $\sum_{n=1}^{\infty} \frac{7}{n^{3/2}}$

Let $f(x) = \frac{7}{x^{3/2}}$. Since f is continuous, decreasing, and positive for $x \geq 1$ and

$$\int_1^\infty \frac{7}{x^{3/2}}dx = 7\int_1^\infty x^{-3/2}dx = -14x^{-1/2}\Big|_1^\infty$$

$$= -\frac{14}{\sqrt{x}}\Big|_1^\infty = 0 + 14 < \infty$$

Then by the integral test the sum must converge.

(ii) $\sum_{n=7}^{\infty} \frac{2n}{n^2+1}$. Let $f(x) = \frac{2x}{x^2+1}$. Since f is continuous, positive and

$$f'(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = \frac{1-2x^2}{(x^2+1)^2} < 0 \quad \text{when } x \geq 7$$

shows f is also decreasing, then

$$\int_7^\infty \frac{2x}{x^2+1}dx = \int_{50}^\infty \frac{1}{u}du = \ln u\Big|_{50}^\infty = \infty$$

$u = x^2+1$, $du = 2x\,dx$

implies by the integral test the sum must diverge.

(iii) $\sum_{n=15}^{\infty} ne^{-n^2}$, Let $f(x) = xe^{-x^2}$. Since f is continuous. Positive and

$$f'(x) = e^{-x^2} + x(-2xe^{-x^2}) = (1-2x^2)e^{-x^2} < 0 \text{ for } x \geq 15$$

shows f is decreasing, then

$$\int_{15}^\infty xe^{-x^2}dx = \frac{1}{2}\int_{225}^\infty e^{-u}du = -\frac{1}{2}e^{-u}\Big|_{225}^\infty = \frac{1}{2}e^{-225} < \infty$$

$u = x^2$ $du = 2x\,dx$

implies by the integral test the sum must converge.