Math 182 Quiz 5 Version B

1. Find the following integrals:

(i) $\int \frac{2x+1}{x^2-4x-5}\,dx = \int \frac{2x+1}{(x-5)(x+1)}\,dx = \int \left(\frac{a}{x-5} + \frac{b}{x+1}\right) dx$

$= a\ln|x-5| + b\ln|x+1| + C$

$= \frac{11}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C$

$a(x+1) + b(x-5) = 2x+1$

$(a+b)x + a - b = 2x+1$ $\qquad a = 2 - \frac{1}{6} = \frac{11}{6}$

$a+b=2 \quad a-5b=1 \quad a=5b+1 \quad 6b+1=2 \quad b=\frac{1}{6}$

(ii) $\int \frac{4\ln(2+4x)}{1+2x}\,dx = 2\int u\,du = u^2 + C$

$u = \ln(2+4x)$

$du = \frac{1}{2+4x}\cdot 4\,dx = \frac{1}{1+2x}\cdot 2\,dx$

$= (\ln(2+4x))^2 + C.$

(iii) $\int_1^{\sqrt{5}} \frac{2w^2}{\sqrt{w^2-1}}\,dw = \int_{\sec^{-1}1}^{\sec^{-1}\sqrt{5}} 2\frac{\sec^2 u \sec u \tan u}{\tan u}\,du$

let $w = \sec u$ $\quad dw = \sec u \tan u\,du$ (since $\sec^2 u - 1 = \tan^2 u$)

$\int \sec^3 u\,du = \sec u \tan u - \int \sec u \tan^2 u\,du$

$p = \sec u \qquad dp = \sec u \tan u\,du$

$dq = \sec^2 u\,du \quad q = \tan u$

$= \sec u \tan u - \int \sec^3 u\,du + \int \sec u\,du$

Thus $2\int_{\sec^{-1}1}^{\sec^{-1}\sqrt{5}} \sec^3 u\,du = \left(\sec u\sqrt{\sec^2 u-1} + \ln|\sec u + \sqrt{\sec^2 u - 1}|\right)\Big|_{\sec^{-1}1}^{\sec^{-1}\sqrt{5}}$

$= \sqrt{5}\sqrt{5-1} + \ln|\sqrt{5}+\sqrt{5-1}| = 2\sqrt{5} + \ln|2+\sqrt{5}|.$

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2. Show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

converges if $p > 1$ and diverges if $p \leq 1$.

Case $p>1$. Then $f(x) = \frac{1}{x^p}$ is positive, decreasing and continuous for $x \geq 1$. Since $p>1$ then $p-1>0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \frac{x^{1-p}}{1-p}\Big|_1^\infty = \frac{-1}{p-1}\cdot\frac{1}{x^{p-1}}\Big|_1^\infty$$

$$= 0 + \frac{1}{p-1} < \infty$$

implies by the integral test that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Case $p=1$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ because this is the harmonic series, which has already been shown to diverge.

Case $0<p<1$. Then $f(x) = \frac{1}{x^p}$ is positive, decreasing and continuous for $x \geq 1$. Since $0<p<1$ then $p-1<0$ and

$$\int_1^\infty \frac{1}{x^p}\,dp = \frac{x^{1-p}}{1-p}\Big|_1^\infty = \infty - \frac{1}{1-p} = \infty$$

implies by the integral test that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

Case $p \leq 0$. Then the general term does not tend to zero, so the series diverges.

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3. Determine whether the following series converge or diverge and explain your answer.

(i) $\sum_{n=1}^{\infty} n^2$

Since $\lim_{n\to\infty} n^2 = \infty$ then the general term does not tend to zero. Therefore the series diverges.

(ii) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{3^n}$ Let $a_n = \frac{n^2+n+1}{3^n}$. Then by the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^2+(n+1)+1}{3^{n+1}}\right)}{\left(\frac{n^2+n+1}{3^n}\right)} = \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^2+(n+1)+1}{n^2+n+1}$$

by L'Hôpital's rule...

$$= \frac{1}{3} \lim_{n\to\infty} \frac{2(n+1)+1}{2n+1} = \frac{1}{3} \lim_{n\to\infty} \frac{2}{2} = \frac{1}{3} < 1$$

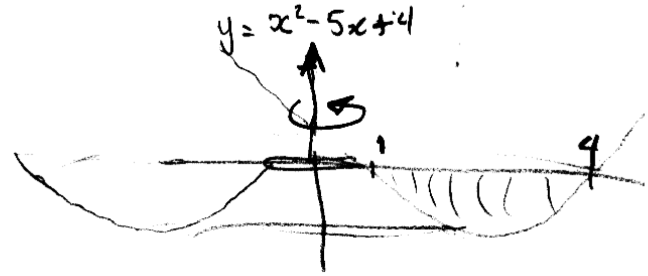
which is less than one, so the series converges.

(iii) $\sum_{n=1}^{\infty} \frac{(n+7)^n}{(2n+1)^n}$. Let $a_n = \left(\frac{n+7}{2n+1}\right)^n$. The by the root test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n+7}{2n+1} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$$

which is less than one, so the series converges.

4. Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 5x + 4$ and $y = 0$ about the y-axis.



$y = x^2 - 5x + 4$
$= (x-4)(x-1)$

<u>Using shell method,</u>

$$V = -2\pi \int_1^4 xy\,dx =$$

$$= -2\pi \int_1^4 x(x^2-5x+4)\,dx = -2\pi \int_1^4 (x^3 - 5x^2 + 4x)\,dx$$

$$= -2\pi\left(\frac{x^4}{4} - \frac{5}{3}x^3 + 2x^2\right)\Bigg|_1^4 = -2\pi x^2\left(\frac{x^2}{4} - \frac{5}{3}x + 2\right)\Bigg|_1^4$$

$$= -32\pi\left(\frac{16}{4} - \frac{5\cdot 4}{3} + 2\right) + 2\pi\left(\frac{1}{4} - \frac{5}{3} + 1\right)$$

$$= -32\pi\left(\frac{18-20}{3}\right) + 2\pi\left(\frac{3-20+12}{12}\right) = \frac{64\pi}{3} - \frac{5\pi}{6} = \frac{123}{6}\pi = \frac{41\pi}{2}$$

5. Find the Taylor polynomial of order 3 generated by $f(x) = \dfrac{1}{1+\sin x}$ at $a = 0$.

$$f'(x) = \frac{-\cos x}{(1+\sin x)^2} = -\cos x\,(1+\sin x)^{-2}$$

$f(0) = 1 \quad f'(0) = -1$
$f''(0) = 2$

$$f''(x) = \sin x(1+\sin x)^{-2} + 2\cos^2 x\,(1+\sin x)^{-3}$$

$$f'''(x) = \cos x(1+\sin x)^{-2} - 2\sin x(1+\sin x)^{-3}\cos x$$
$$- 4\cos x\sin x(1+\sin x)^{-3} - 6\cos^3 x\,(1+\sin x)^{-4}$$

$$f'''(0) = 1 - 6 = -5$$

The polynomial is

$$f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0)$$

$$= 1 - x + x^2 - \frac{5}{6}x^3$$