Differentiation of The Remainder Term

Example:

$$\frac{d}{dx}\int_0^x \frac{(x-t)^{2n+2}}{(2n+2)!}\cos t\,dt$$

One could use the fundamental theorem of Calculus, except there is another x inside the integrand. Thus we'll use the definition of derivative.

$$\frac{d}{dx}\int_0^x \frac{(x-t)^{2n+2}}{(2n+2)!}\cos t\,dt = \lim_{w\to x}\frac{\int_0^w \frac{(w-t)^{2n+2}}{(2n+2)!}\cos t\,dt - \int_0^x \frac{(x-t)^{2n+2}}{(2n+2)!}\cos t\,dt}{w-x}$$

Now introduce the intermediate point of comparison

$$\int_0^x \frac{(w-t)^{2n+2}}{(2n+2)!}\cos t\,dt$$

into the numerator to obtain.

$$= \lim_{w\to x}\frac{\int_0^w \frac{(w-t)^{2n+2}}{(2n+2)!}\cos t\,dt - \int_0^x \frac{(w-t)^{2n+2}}{(2n+2)!}\cos t\,dt}{w-x}$$

$$+ \lim_{w\to x}\frac{\int_0^x \frac{(w-t)^{2n+2}}{(2n+2)!}\cos t\,dt - \int_0^x \frac{(x-t)}{(2n+2)!}\cos t\,dt}{w-x}$$

$$= \lim_{w\to x}\frac{1}{w-x}\int_x^w \frac{(w-t)^{2n+2}}{(2n+2)!}\cos t\,dt$$

$$+ \lim_{w\to x}\int_0^x \frac{(w-t)^{2n+2}-(x-t)^{2n+2}}{w-x}\,\frac{\cos t}{(2n+2)!}\,dt$$

The first limit we'll treat using the intermediate value theorem for integrals.

$$\lim_{w \to x} \frac{1}{w-x} \int_x^w \frac{(w-t)^{2n+2}}{(2n+2)!} \cos t \, dt$$

$$= \lim_{w \to x} \frac{(w-\xi)^{2n+2}}{(2n+2)!} \cos \xi \quad \text{where } \xi \text{ is between } x \text{ and } w.$$

Since ξ is between x and w then

$$\xi \to x \quad \text{as} \quad w \to x$$

Therefore,

$$\cdots = \frac{(x-x)^{2n+2}}{(2n+2)!} \cos x = 0.$$

Now we compute the second limit.

$$\lim_{w \to x} \int_0^x \frac{(w-t)^{2n+2} - (x-t)^{2n+2}}{w-x} \frac{\cos t}{(2n+2)!} \, dt$$

Notice that the fraction inside the integral converges to the derivative of $(x-t)^{2n+2}$ as $w \to x$. Since

$$\frac{d}{dx}(x-t)^{2n+2} = (2n+2)(x-t)^{2n+1}$$

then

$$\cdots = \int_0^x (2n+2)(x-t)^{2n+1} \frac{\cos t}{(2n+2)!} \, dt = \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} \cos t \, dt.$$

Therefore

$$\frac{d}{dx} \int_0^x \frac{(x-t)^{2n+2}}{(2n+2)!} \cos t \, dt = \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} \cos t \, dt.$$