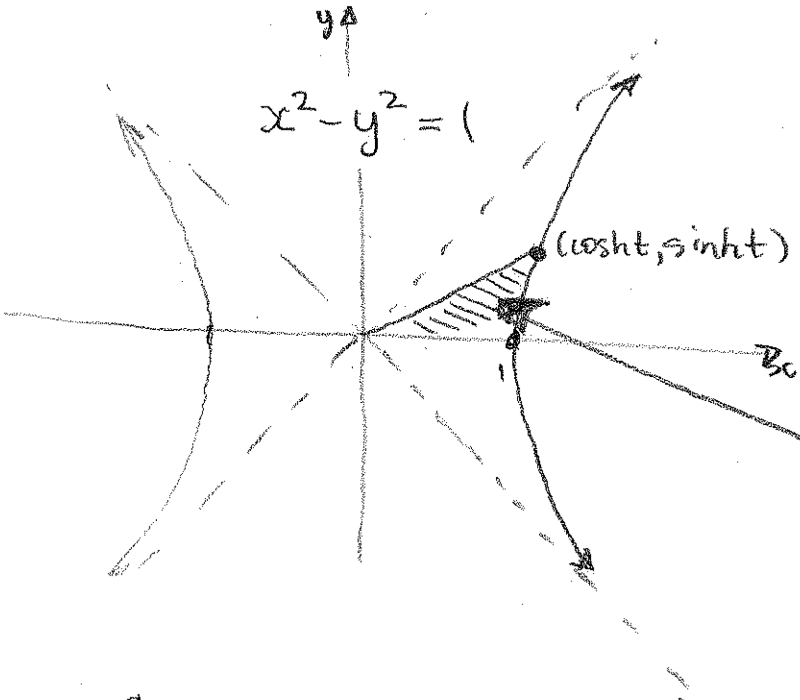
Hyperbolic Trigonometric Functions.



The functions cosht and sinht are defined so that this point on the hyperbola determines an area where

$$A = \frac{t}{2}$$

Define $\tanh t = \dfrac{\cosh t}{\sinh t}$

Coördinate transform to make this hyperbola look like the one from last semester:

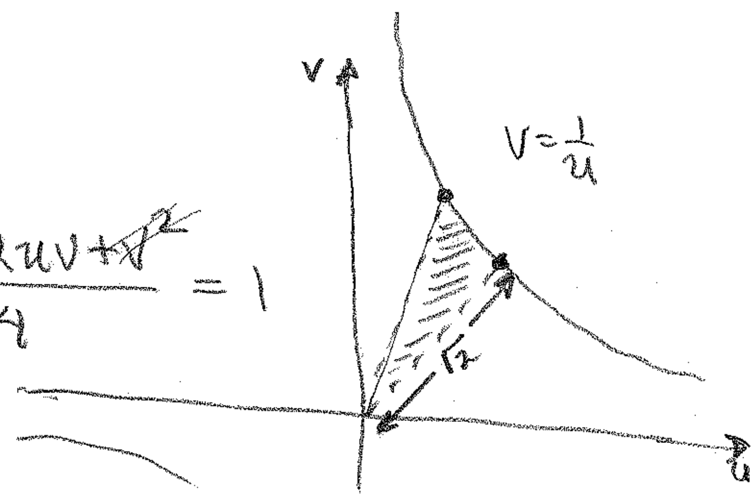
$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

Then the hyperbola becomes

$$\left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2 = 1$$

$$\frac{u^2+2uv\cancel{+v^2}}{4} - \frac{\cancel{u^2}-2uv\cancel{+v^2}}{4} = 1$$

or $v = \dfrac{1}{u}$



Thus

$$x = \cosh t = \frac{u+u^{-1}}{2} \text{ and } y = \sinh t = \frac{u-u^{-1}}{2}.$$

Note: this coordinate transform was actually a 45° rotation with a $\sqrt{2}$ magnification.

We now compute the area A using calculus:

$$A = \int_0^{\cosh t} x \tanh t \, dx - \int_1^{\cosh t} \sqrt{x^2-1}\, dx$$

Compute the first integral.

$$\int_0^{\cosh t} x \tanh t \, dx = \frac{x^2}{2}\tanh t \Big|_0^{\cosh t}$$

$$= \frac{\cosh^2 t \tanh t}{2} = \frac{\cosh t \sinh t}{2}.$$

For the second integral, first find the antiderivative.

$$\int \sqrt{x^2-1}\, dx = \int \sqrt{\left(\frac{u+u^{-1}}{2}\right)^2 - 1} \cdot \frac{1-u^{-2}}{2}\, du$$

$$x = \frac{u+u^{-1}}{2} \qquad dx = \frac{1-u^{-2}}{2}\, du$$

$$= \int \sqrt{\frac{u^2+2uu^{-1}+u^{-2}}{4} - 1} \cdot \frac{1-u^{-2}}{2}\, du$$

$$= \int \sqrt{\frac{u^2-2+u^{-2}}{4}} \cdot \frac{1-u^{-2}}{2}\, du$$

$$= \int \sqrt{\left(\frac{u-u^{-1}}{2}\right)^2} \cdot \frac{1-u^{-2}}{2}\, du$$

$$= \frac{1}{4}\int (u-u^{-1})\cdot(1-u^{-2})\, du.$$

Continued ...

$$= \frac{1}{4}\int (u - 2u^{-1} + u^{-3})\,du$$

$$= \frac{1}{4}\left(\frac{u^2}{2} - 2\ln u - \frac{u^{-2}}{2}\right)$$

Now, solve for u in terms of x.

$$2x = u + \frac{1}{u}$$

$$2xu = u^2 + 1$$

$$u^2 - 2xu + 1 = 0$$

$$a = 1,\quad b = -2x,\quad c = 1$$

$$u = \frac{2x \pm \sqrt{4x^2 - 1}}{2} = x + \sqrt{x^2 - 1}$$

Therefore

$$\int \sqrt{x^2-1}\,dx = \frac{1}{4}\left(\frac{(x+\sqrt{x^2-1})^2}{2} - 2\ln(x+\sqrt{x^2-1}) - \frac{1}{2(x+\sqrt{x^2-1})^2}\right)$$

Use this to evaluate the definite integral.

$$\int_1^{\cosh t} \sqrt{x^2-1}\,dx = F(\cosh t) - F(1)$$

where

$$F(x) = \frac{1}{4}\left(\frac{(x+\sqrt{x^2-1})^2}{2} - 2\ln(x+\sqrt{x^2-1}) - \frac{1}{2(x+\sqrt{x^2-1})^2}\right)$$

Compute

$$F(1) = \frac{1}{4}\left(\frac{(1+\sqrt{1^2-1})^2}{2} - 2\ln(1+\sqrt{1^2-1}) - \frac{1}{2(1+\sqrt{1^2-1})^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{2} - 2\ln 1 - \frac{1}{2}\right) = 0$$

and

$$x+\sqrt{x^2-1}\,\Big|_{x=\cosh t} = \cosh t + \sqrt{\cosh^2 t - 1} = \cosh t + \sinh t$$

Therefore

$$\frac{t}{2} = A = \frac{\cosh t \sinh t}{2} - F(\cosh t) + F(1)$$

$$= \frac{\cosh t \sinh t}{2} - \frac{(\cosh t + \sinh t)^2}{8}$$

$$+ \frac{1}{2}\ln(\cosh t + \sinh t) + \frac{1}{8(\cosh t + \sinh t)^2}$$

It will now be shown that all the terms on the right hand side of this equation cancel except the one with the logarithm.

Recall $\cosh t = \frac{u+u^{-1}}{2}$ and $\sinh t = \frac{u-u^{-1}}{2}$.

Then

$$\cosh t + \sinh t = \frac{u+u^{-1}}{2} + \frac{u-u^{-1}}{2} = u.$$

and

$$\cosh t \sinh t = \frac{u+u^{-1}}{2} \cdot \frac{u-u^{-1}}{2} = \frac{1}{4}u^2 - \frac{1}{4}u^{-2}.$$

It follows that

$$\frac{\cosh t \sinh t}{2} - \frac{(\cosh t + \sinh t)^2}{8} + \frac{1}{8(\cosh t + \sinh t)^2}$$

$$= \frac{1}{8}u^2 - \frac{1}{8}u^{-2} - \frac{u^2}{8} + \frac{1}{8u^2} = 0$$

Therefore

$$t = \ln(\cosh t + \sinh t)$$

or

$$\cosh t + \sinh t = u = e^t.$$

Consequently

$$\cosh t = \frac{u+u^{-1}}{2} = \frac{e^t + e^{-t}}{2}$$

and

$$\sinh t = \frac{u-u^{-1}}{2} = \frac{e^t - e^{-t}}{2}.$$