# Math 182 Homework 7

#1(8). $\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{x^2-x+6}{x(x^2+3)}\,dx = \int\left(\frac{a}{x} + \frac{bx+c}{x^2+3}\right)dx$

$= a\ln x + \frac{b}{2}\ln(x^2+3) + \frac{c}{3}\int \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2}$

Now $u = x/\sqrt{3}$ so $x = u\sqrt{3}$, $dx = \sqrt{3}\,du$

$$\int \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2} = \int \frac{\sqrt{3}\,du}{1+u^2} = \sqrt{3}\arctan u.$$

so

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = a\ln x + \frac{b}{2}\ln(x^2+3) + \frac{c}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}$$

solve for $a, b$ and $c$.

$$\frac{a}{x} + \frac{bx+c}{x^2+3} = \frac{x^2-x+6}{x(x^2+3)}$$

$a(x^2+3) + bx^2 + cx = x^2 - x + 6$

$(a+b)x^2 + cx + 3a = x^2 - x + 6$

$a+b=1,\quad c=-1,\quad 3a=6$

$a=2,\quad b=-1,\quad c=-1$

Therefore

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = 2\ln x - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}$$

Math 182 Homework 7

#1(ii) $\int \tan^3 z\, dz = \int \tan z \cdot \tan^2 z\, dz$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$= \int \tan z\,(\sec^2 z - 1)\, dz$$

$$\frac{d}{dz}\tan z = \frac{d}{dz}\frac{\sin z}{\cos z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z$$

Let $u = \tan z$ $\quad du = \sec^2 z\, dz$

$$\int \tan z \sec^2 z\, dz = \int u\, du = \frac{u^2}{2} = \frac{\tan^2 z}{2}$$

Let $v = \cos z$, $\quad dv = -\sin z\, dz$

$$\int \tan z\, dz = \int \frac{\sin z}{\cos z}\, dz = \int -\frac{1}{v}\, dv = -\ln v = -\ln \cos z$$

Therefore

$$\int \tan^3 z\, dz = \frac{\tan^2 z}{2} + \ln \cos z .$$

# Math 182 Homework 7

#1 (iii)

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \int_0^1 y e^{-2y}\,dy$$

$u = y \qquad du = dy$

$dv = e^{-2y}dy \qquad v = -\frac{1}{2}e^{-2y}$

$$\ldots = -\frac{y}{2}e^{-2y}\Big|_0^1 + \int_0^1 \frac{1}{2}e^{-2y}\,dy$$

$$= -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2y}\Big|_0^1$$

$$= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4} = \frac{1}{4} - \frac{3}{4e^2}$$

Math 182 Homework 2

#2 For $t > 0$ define

$$G(t) = \int_1^t \frac{1}{x^2\sqrt{x^2+1}}\,dx.$$

(i) Solve using the substitution $u = \frac{1}{x^2}$.

$$x^2 = \frac{1}{u}, \quad x = \frac{1}{\sqrt{u}}, \quad dx = -\frac{1}{2}u^{-3/2}\,du$$

$$G(t) = \int_1^{1/t^2} \frac{u}{\sqrt{1+\frac{1}{u}}} \cdot \frac{-1}{2} u^{-3/2}\,du$$

$$= -\frac{1}{2}\int_1^{1/t^2} \frac{1}{\sqrt{u+1}}\,du$$

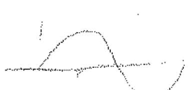
$$= -\sqrt{u+1}\,\Big|_1^{1/t^2} = \sqrt{2} - \sqrt{\frac{1}{t^2}+1}$$

(ii) Solve using the substitution $w = \arctan x$.

$$x = \tan w, \quad dx = \sec^2 w\,dw$$

$$G(t) = \int_{\pi/4}^{\arctan t} \frac{\sec^2 w}{\tan^2 w\sqrt{\tan^2 w+1}}\,dw = \int_{\pi/4}^{\arctan t} \frac{\sec w}{\tan^2 w}\,dw$$

$$= \int_{\pi/4}^{\arctan t} \frac{1}{\cos w} \cdot \frac{\cos^2 w}{\sin^2 w}\,dw = \int_{\pi/4}^{\arctan t} \frac{\cos w}{\sin^2 w}\,dw$$



$$v = \sin w, \quad dv = \cos w\,dw$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin(\arctan t) = \frac{t}{\sqrt{1+t^2}}$$

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$$\sin\vartheta = \cos\vartheta \tan\vartheta = \pm\sqrt{1-\sin^2\vartheta}\ \tan\vartheta$$

$$\sin^2\vartheta = (1-\sin^2\vartheta)\tan^2\vartheta$$

$$\sin^2\vartheta(1+\tan^2\vartheta) = \tan^2\vartheta$$

$$\sin^2\vartheta = \frac{\tan^2\vartheta}{1+\tan^2\vartheta}$$

Therefore, if $\vartheta \in [0,\pi]$ we have

$$\sin\vartheta = \frac{\tan\vartheta}{\sqrt{1+\tan^2\vartheta}}$$

Let $\vartheta = \arctan t$. Then $\vartheta \in [0,\pi]$ and so

$$\sin\arctan x = \frac{\tan(\arctan t)}{\sqrt{1+\tan^2\arctan t}} = \frac{t}{\sqrt{1+t^2}}$$

Now

$$G(t) = \int_{1/\sqrt{2}}^{t/\sqrt{1+t^2}} \frac{1}{v^2}\,dv = -\frac{1}{v}\Big|_{1/\sqrt{2}}^{t/\sqrt{1+t^2}}$$

$$= \sqrt{2} - \frac{\sqrt{1+t^2}}{t} = \sqrt{2} - \sqrt{\frac{1}{t^2}+1}$$

So the methods give the same answers.

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#2 (iii) Find the limits $\lim_{t\to\infty} G(t)$ and $\lim_{t\to 0^+} G(t)$.

$$\lim_{t\to\infty} G(t) = \lim_{t\to\infty} \sqrt{2} - \sqrt{\frac{1}{t^2}+1} = \sqrt{2} - 1$$

$$\lim_{t\to 0^+} G(t) = \lim_{t\to 0^+} \sqrt{2} - \sqrt{\frac{1}{t^2}+1} = \sqrt{2} - \sqrt{\lim_{t\to 0^+} \frac{1}{t^2} + 1}$$

$$= \sqrt{2} - \sqrt{\infty + 1} = -\infty$$

Math 182 Homework 7

\#31 Consider the curve $(f(t), g(t))$ given by

$f(t) = e^t - t$, $g(t) = 4e^{t/2}$ where $-8 \le t \le 3$.

(i) Find the length of this curve.

$f'(t) = e^t - 1$, $g'(t) = 2e^{t/2}$

$$L = \int_{-8}^{3} \sqrt{f'(t)^2 + g'(t)^2}\, dt = \int_{-8}^{3} \sqrt{(e^t - 1)^2 + 4e^t}\, dt$$

$$= \int_{-8}^{3} \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= \int_{-8}^{3} \sqrt{e^{2t} + 2e^t + 1}\, dt = \int_{-8}^{3} (e^t + 1)\, dt$$

$$= e^t + t \Big|_{-8}^{3} = e^3 - e^{-8} + 3 + 8$$

$$= e^3 - e^{-8} + 11$$

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#3(1) Find the equation of the line tangent to this curve at the point (1,4)

$f(t) = e^t - t = 1$

$g(t) = 4e^{t/2} = 4$ $\qquad e^{t/2} = 1$

$t/2 = \ln 1 = 0$

so $t = 0$

Plug in $t=0$ to $f(0)$, $g(0)$, $f'(0)$ and $g'(0)$

$f(0) = e^0 - 0 = 1$

$g(0) = 4e^{0/2} = 4$

$f'(0) = e^0 - 1 = 1 - 1 = 0$

$g'(0) = 2e^{t/2} = 2$

Equation of the tangent line

$(y - g(0))f'(0) = (x - f(0))g'(0)$

$(y-4)\cdot 0 = (x-1)\cdot 2$

$x = 1$

Math 182 Homework 7

#3 (iii) Find the equation of the circle osculating with this curve at the point (1,4).

$f''(t) = e^t$ $\qquad$ $f''(0) = 1$

$g''(t) = e^{t/2}$ $\qquad$ $g''(0) = 1$

Curvature

$$K = \frac{g''(0)f'(0) - g'(0)f''(0)}{(f'(0)^2 + g'(0)^2)^{3/2}} = \frac{1\cdot 0 - 2\cdot 1}{(0^2 + 2^2)^{3/2}} =$$

$$= \frac{-2}{2^3} = -\frac{1}{4}$$

Radius of curvature

$\rho = 4$

Normal to curve

$$N = \left(\frac{-g'(0)}{\sqrt{f'(0)^2 + g'(0)^2}}, \frac{f'(0)}{\sqrt{f'(0)^2 + g'(0)^2}}\right) = \left(\frac{-2}{2}, 0\right) = (-1, 0)$$

Center of circle

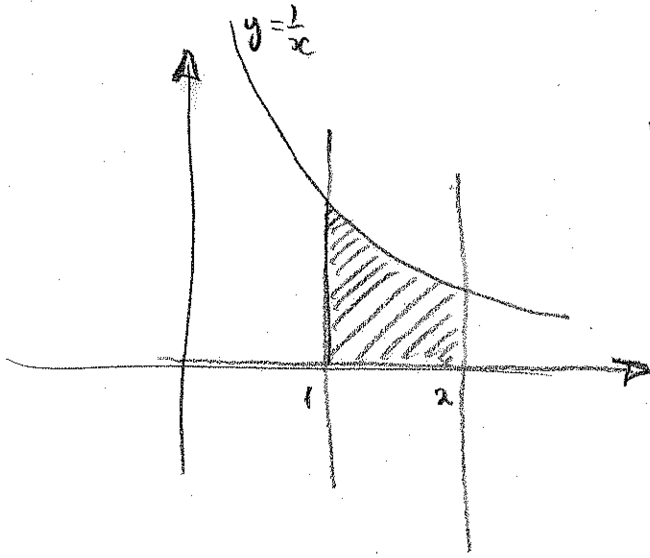
$$(x_0, y_0) - \rho N = (1,4) - 4(-1,0) = (5,4)$$

Equation of circle

$$(x-5)^2 + (y-4)^2 = 16.$$

Math 182 Homework 7

#4. Find the volume generated by revolving the region bounded by $y = 1/x$, $x=1$, $x=2$ and $y=0$ about $x$-axis.




$$V = \int_1^2 \pi f(x)^2 \, dx$$

$$= \int_1^2 \pi \frac{1}{x^2} \, dx$$

$$= -\frac{\pi}{x} \Big|_1^2$$

$$= \pi \left(1 - \frac{1}{2}\right)$$

$$= \frac{\pi}{2}.$$

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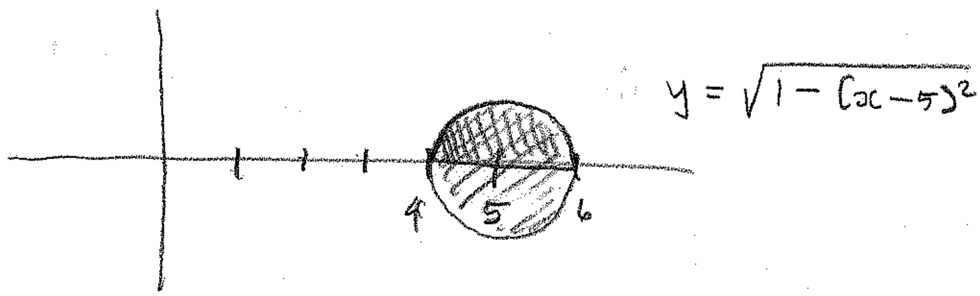
#5. Consider the circle given by $x^2+24+y^2=10x$. Find the volume generated by revolving the region bounded by this circle about the y-axis.

Find center of the circle:

$$x^2-10x+24+y^2=0$$

$$(x-5)^2-25+24+y^2=0$$

$$(x-5)^2+y^2=1$$




We compute the volume generated by the top half of the circle and the double it

$$V=2\int_4^6 2\pi x f(x)\,dx = 4\pi\int_4^6 x\sqrt{1-(x-5)^2}\,dx$$

$$u=x-5 \qquad du=dx$$

$$=4\pi\int_{-1}^{1}(u+5)\sqrt{1+u^2}\,du$$

#5 continues....

$$\int_{-1}^{1} u\sqrt{1+u^2}\,du = \frac{1}{3}(1+u^2)^{3/2}\Big|_{-1}^{1} = 0$$

$$\int_{-1}^{1} \sqrt{1-u^2}\,du = 2\int_{0}^{1} \sqrt{1-u^2}\,du$$

$u = \sin\theta \qquad du = \cos\theta\, d\theta$

$$= 2\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\, d\theta = 2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \cos 2\theta\, d\theta = \frac{\pi}{2} + \frac{\sin 2\theta}{2}\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{\sin\pi}{2} - \frac{\sin 0}{2} = \frac{\pi}{2}$$

Therefore

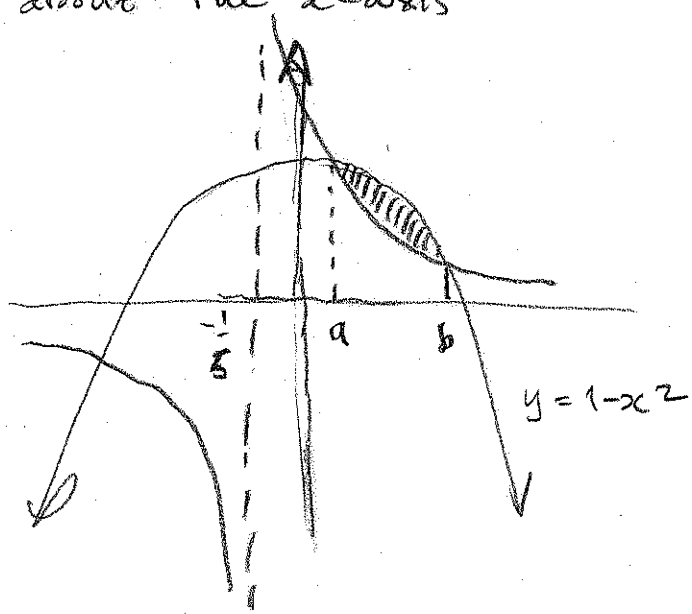
$$V = 4\pi\int_{-1}^{1} (u+5)\sqrt{1+u^2}\,du = 10\pi^2$$

Math 182 Homework 7

#6 Find to 5 digit accuracy the volume generated by revolving the region bounded by the curves

$$y = \frac{2}{1+5x} \quad \text{and} \quad y = 1-x^2$$

about the $x$-axis




$$V = \int_a^b \pi (1-x^2)^2\,dx - \int_a^b \pi \left(\frac{2}{1-5x}\right)^2 dx$$

Now

$$\int_a^b (1-x^2)^2\,dx = \int_a^b (x^4 - 2x^2 + 1)\,dx = \left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)\Bigg|_a^b$$

and

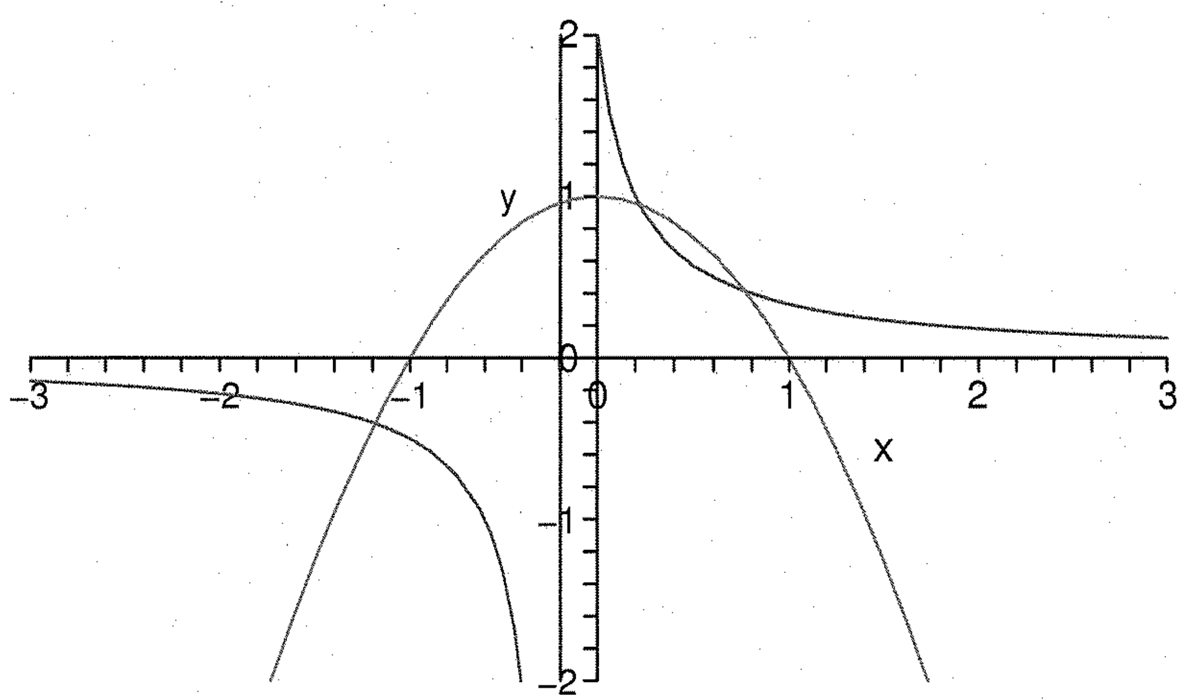
$$\int_a^b \frac{1}{(1+5x)^2} = -\frac{1}{5}\,\frac{1}{1+5x}\Bigg|_a^b$$

```
> restart;
> # Math 182 Homework 7
  #
  # Question #7 continues...
> f:=2/(1+5*x);
  g:=1-x^2;
```

$$f := \frac{2}{1 + 5\,x}$$

$$g := 1 - x^2$$

```
> plot([f,g],x=-3..3,y=-2..2);
```




```
> a:=fsolve(f=g,x=0.2);
```

$$a := 0.2204279572$$

```
> b:=fsolve(f=g,x=1);
```

$$b := 0.7652427560$$

```
> V:=int(Pi*(g^2-f^2),x=a..b);
```

$$V := 0.2852020332$$

```
>
```

Math 182 Homework #7

#6 continues...

Using Maple (see previous page), we find that $a \approx 0.220428$

$b \approx 0.765243$

Therefore

$$V = \pi\left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)\Bigg|_a^b + \pi\,\frac{4}{5}\cdot\frac{1}{1+5x}\Bigg|_a^b$$

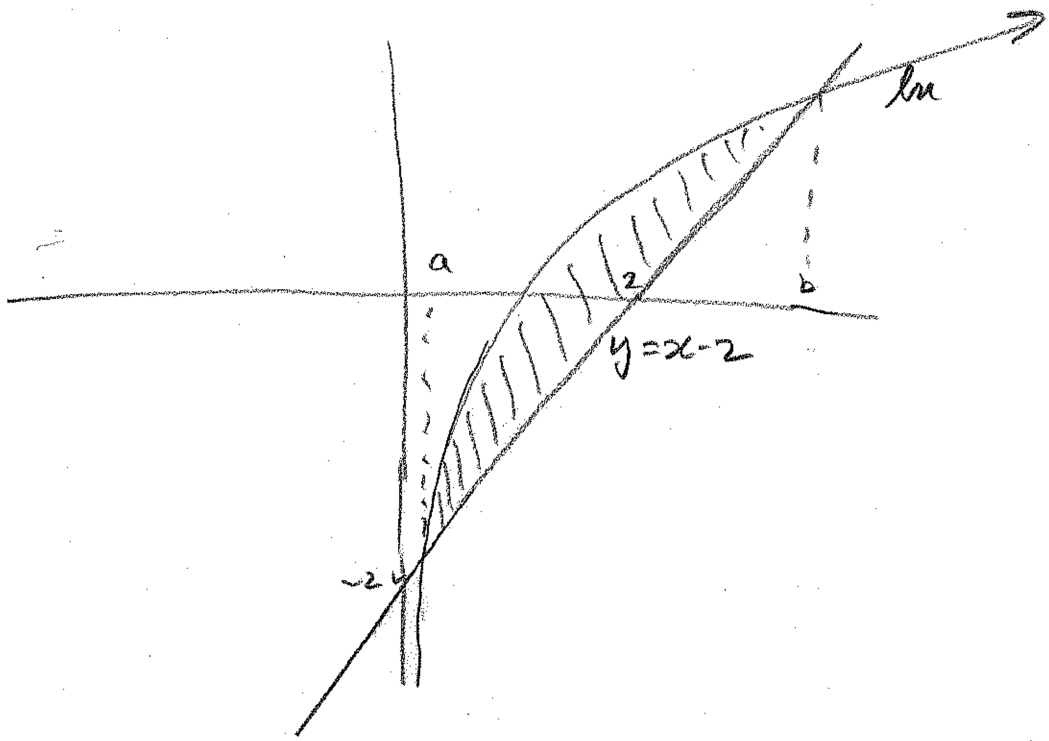
$\approx 0.285202.$

Math 182 Homework 7

\#7 Find to 5 digit accuracy the volume generated by revolving the region bounded by the curves

$$y = \ln x \quad \text{and} \quad y = x - 2$$

about the y-axis.




$$V = \int_a^b 2\pi x \ln x \, dx - \int_a^b 2\pi x (x-2)\, dx$$

Now

$$\int_a^b x(x-2)\,dx = \frac{x^3}{3} - x^2 \Big|_a^b$$

and

$$\int_a^b x \ln x \, dx = \frac{x^2}{2}\ln x \Big|_a^b - \int_a^b \frac{x}{2}\,dx = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_a^b$$

$u = \ln x \quad du = \frac{1}{x}dx$

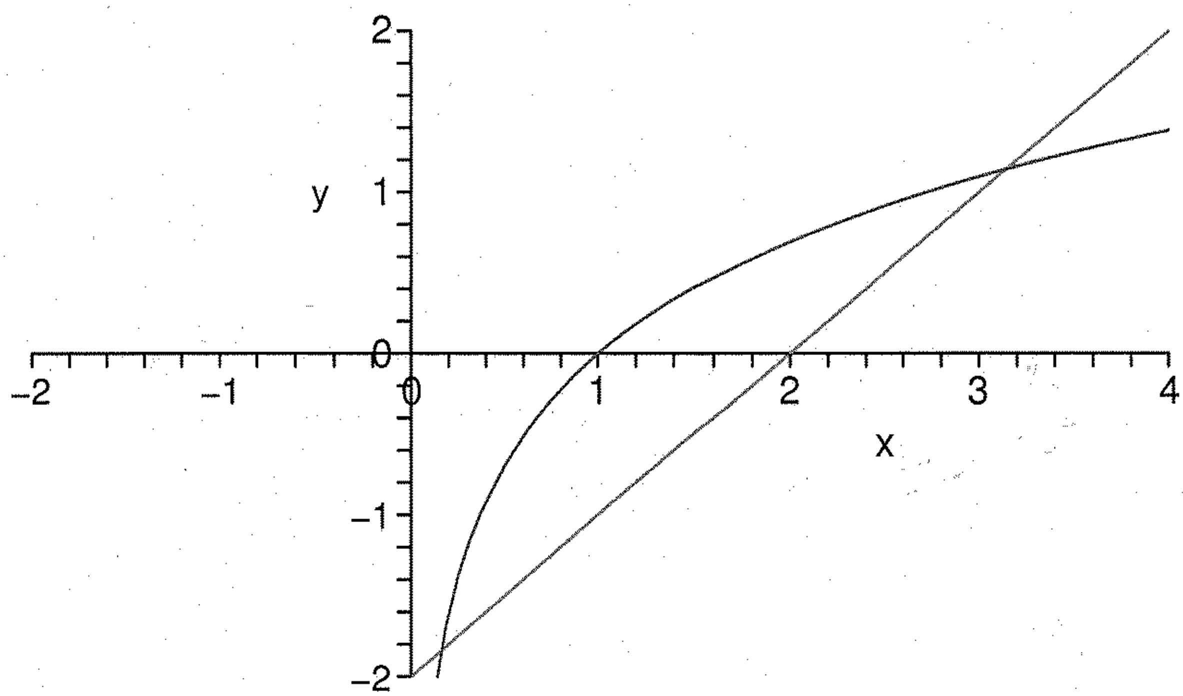
$dv = x\,dx \quad v = \frac{x^2}{2}$

```
> restart;
> # Math 182 Homework 7
  #
  # Question #7 continues...
> f:=log(x);
  g:=x-2;
```

$$f := \ln(x)$$

$$g := x - 2$$

```
> plot([f,g],x=-2..4,y=-2..2);
```




```
> a:=fsolve(f=g,x=0.2);
```

$$a := 0.1585943396$$

```
> b:=fsolve(f=g,x=2);
```

$$b := 3.146193221$$

```
> V:=int(2*Pi*x*(f-g),x=a..b);
```

$$V := 17.09930676$$

```
>
```

Math 182 Homework 7

#7 continues...

Using Maple (see previous page), we find that

$a \approx 0.158594$

$b \approx 3.146193$

Therefore

$$V = 2\pi\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Bigg|_a^b - \left(\frac{x^3}{3} - x^2\right)\Bigg|_a^b$$

$\approx 17.0993$

Math 182 Homework 7

#8 Let $f(x) = \frac{1}{\sqrt{x}}$.

(i) Find $f'(x)$, $f''(x)$ and $f'''(x)$.

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = \frac{1}{2} \cdot \left(\frac{-1}{2}\right) x^{-3/2} = -\frac{1}{4} x^{-3/2}$

$f'''(x) = -\frac{1}{4} \cdot \frac{-3}{2} x^{-5/2} = \frac{3}{8} x^{-5/2}$

(ii) Use induction to show that the nth derivative of f satisfies the formula

$$f^{(n)}(x) = \frac{(-1)^n}{4^n} \frac{(2n)!}{n!} x^{-\frac{2n+1}{2}}$$

Base case: $n=1$ then

$$f^{(1)}(x) = \frac{-1^1}{4^1} \frac{2!}{1!} x^{-\frac{2+1}{2}} = -\frac{1}{2} x^{-1/2}$$

which agrees with the computation in part (i).

Induction step:

Suppose $f^{(n)}(x) = \frac{(-1)^n}{4^n} \frac{(2n)!}{n!} x^{-\frac{2n+1}{2}}$

Then $f^{(n+1)}(x) = \frac{(-1)^n}{4^n} \frac{(2n)!}{n!} \cdot -\frac{2n+1}{2} x^{-\frac{2n+1}{2}-1}$

Math 182 Homework 7

#8 (ii) Continues...

$$f^{(n+1)}(x) = \frac{(-1)^{n+1}}{4^n} \cdot \frac{(2n)!}{n!} \cdot \frac{2n+1}{2} \cdot x^{-\frac{2(n+1)+1}{2}}$$

$$= \frac{(-1)^{n+1}}{4^{n+1}} 4 \frac{(2(n+1))!}{(n+1)!} \frac{n+1}{\cancel{(2n+1)}(2n+2)} \cdot \frac{\cancel{2n+1}}{2} x^{-\frac{2(n+1)+1}{2}}$$

$$= \frac{(-1)^{n+1}}{4^{n+1}} \frac{(2(n+1))!}{(n+1)!} \cancel{\frac{4(n+1)}{(2n+2)\cdot 2}} x^{-\frac{2(n+1)+1}{2}}$$

$$= \frac{(-1)^{n+1}}{4^{n+1}} \frac{(2(n+1))!}{(n+1)!} x^{-\frac{2(n+1)+1}{2}}$$

which finishes the induction.