Honors Math 182 Homework 9 Version A

1. Taylor's formula for $f(x) = \ln(1-x)$ when $a = 0$ is

$$\ln(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} - \int_0^x \frac{t^n}{1-t}\,dt$$

(i) Given $|x| < 1$, the remainder may be estimated as

$$\left|\int_0^x \frac{t^n}{1-t}\,dt\right| \le \left|\int_0^x \frac{|t|^n}{1-|x|}\,dt\right| = \frac{1}{1-|x|}\int_0^{|x|} t^n\,dt = \frac{|x|^{n+1}}{(1-|x|)(n+1)}.$$

Use this estimate to show that if $|x| < 1$ then the remainder term tends to zero as $n \to \infty$.

(ii) Given $x = 1/2$ estimate how large n needs to be so the bound on the remainder is less than 0.5×10^{-4}.

2. Use Taylor series or L'Hôpital's rule to find the following limits if they exist.

(i) $\displaystyle \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$

(ii) $\displaystyle \lim_{x\to 0^+} \frac{x^6}{\log(1+x^2) - x^2\cos x}$

(iii) $\displaystyle \lim_{x\to 0} \frac{\sin 3x}{x^3}$

(iv) $\displaystyle \lim_{x\to 0^-} \frac{\sin x - x\cos x}{x^3}$

(v) $\displaystyle \lim_{x\to 0} \frac{\ln(1-x^2) + x\arctan x}{x^4}$

(vi) $\displaystyle \lim_{x\to 0^-} \frac{1-\cos x}{\sin x}$

(vii) $\displaystyle \lim_{x\to 0} \frac{e^x - \sin x}{\cos x}$

(viii) $\displaystyle \lim_{x\to 0^+} \frac{xe^{-x^2} - \sin x}{2xe^x + \ln(1-2x)}$

(ix) $\displaystyle \lim_{x\to 0} \frac{1-\sqrt{1+4x^2}}{\sin^4 x}$

(x) $\displaystyle \lim_{x\to 0} \frac{x + (x-1)\ln(x+1)}{xe^x - \sin x}$