Honors Math 182 Homework 9

1. Taylors formula for $f(x) = \ln(1-x)$ when $a=0$ is

$$\ln(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} - \int_0^x \frac{t^n}{1-t}\,dt$$

(i) Given $|x|<1$, the remainder may be estimated

$$\left|\int_0^x \frac{t^n}{1-t}\,dt\right| \le \left|\int_0^x \frac{|t|^n}{1-|x|}\,dt\right|$$

$$= \frac{1}{1-|x|}\int_0^{|x|} t^n\,dt = \frac{|x|^{n+1}}{(1-|x|)(n+1)}$$

Use this estimate to show that if $|x|<1$ then the remainder term tends to zero as $n \to \infty$

$$0 \le |R_n| \le \frac{|x|^{n+1}}{(1-|x|)(n+1)} \le \frac{1}{1-|x|}\cdot\frac{1}{n+1} \to 0$$

as $n \to \infty$.

(ii) Given $x = 1/2$ estimate how large n needs to be so the bound on the remainder is less that 0.5×10^{-4}.

n	$\frac{1}{1-\frac{1}{2}}\cdot\left(\frac{1}{2}\right)^{n+1}\cdot\frac{1}{n+1}$
8	0.000434
9	0.000195
10	0.000089
11	0.000041 < 0.00005

Therefore $n \ge 11$ ensures the remainder is less than 0.5×10^{-4}.

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2(i) $\displaystyle\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2}$

$e^u = 1+u+O(u^2)$ as $u \to 0$

$e^{x^2} = 1+x^2+O(x^4)$ as $x\to 0$

$\cos x = 1-\frac{x^2}{2}+O(x^4)$

$$\therefore = \lim_{x\to 0} \frac{\cancel{1}+x^2+O(x^4)-\left(\cancel{1}-\frac{x^2}{2}+O(x^4)\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{3}{2}x^2+O(x^4)}{x^2} = \lim_{x\to 0} \frac{3}{2}+O(x^2)$$

$$= \frac{3}{2}$$

2(ii) $\displaystyle\lim_{x\to 0^+} \frac{x^6}{\log(1+x^2)-x^2\cos x}$

$\log(1-u) = -u-\frac{u^2}{2}-\frac{u^3}{3}+O(u^4)$ as $u\to 0$

$\log(1+x^2) = \cancel{x^2}-\cancel{\frac{x^4}{2}}+\frac{x^6}{3}+O(x^8)$ as $x\to 0$

$x^2\cos x = x^2\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+O(x^6)\right)$

$= \cancel{x^2}-\cancel{\frac{x^4}{2}}+\frac{x^6}{24}+O(x^8)$

2(ii) continues...

$$\lim_{x\to 0^+} \frac{x^6}{\log(1+x^2)-x^2\cos x}$$

$$= \lim_{x\to 0^+} \frac{x^6}{\frac{x^6}{3}+O(x^8)-\frac{x^6}{24}}$$

$$= \lim_{x\to 0^+} \frac{1}{\frac{7}{24}+O(x^2)}$$

$$= \frac{1}{\lim_{x\to 0^+} \frac{7}{24}+O(x^2)} = \frac{24}{7}$$

2(iii) $\lim_{x\to 0} \frac{\sin 3x}{x^3} = \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)\left(\lim_{x\to 0} \frac{1}{x^2}\right)$

$\sin u = u - O(u^3)$ as $u \to 0$

$\sin 3x = 3x - O(x^3)$ as $x \to 0$

$$\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{3x-O(x^3)}{x} = \lim_{x\to 0} 3 - O(x^2) = 3$$

Therefore

$$\lim_{x\to 0} \frac{\sin 3x}{x^3} = 3\cdot\infty = \infty.$$

2(iv) $\displaystyle \lim_{x \to 0^-} \frac{\sin x - x\cos x}{x^3}$

$$\sin x = \cancel{x} - \frac{x^3}{6} + O(x^5) \qquad \text{as } x \to 0$$

$$x\cos x = x\left(1 - \frac{x^2}{2} + O(x^4)\right) = \cancel{x} - \frac{x^3}{2} + O(x^5)$$

$$\therefore = \lim_{x \to 0^-} \frac{-\frac{x^3}{6} + \frac{x^3}{2} + O(x^5)}{x^3}$$

$$= \lim_{x \to 0^-} \frac{1}{3} + O(x^2) = \frac{1}{3}$$

2(v) $\displaystyle \lim_{x \to 0} \frac{\ln(1-x^2) + x\arctan x}{x^4}$

$$\ln(1-u) = -u - \frac{u^2}{2} + O(u^3) \qquad \text{as } u \to 0$$

$$\ln(1-x^2) = \cancel{-x^2} - \frac{x^4}{2} + O(x^6) \qquad \text{as } x \to 0$$

$$x\arctan x = x\left(x - \frac{x^3}{3} + O(x^5)\right)$$

$$= \cancel{x^2} - \frac{x^4}{3} + O(x^6)$$

$$\therefore = \lim_{x \to 0} \frac{-\frac{x^4}{2} - \frac{x^4}{3} + O(x^6)}{x^4}$$

$$= \lim_{x \to 0} -\frac{5}{6} + O(x^2) = -\frac{5}{6}.$$

2(vi) $\lim_{x\to 0^-} \frac{1-\cos x}{\sin x} = \lim_{x\to 0^-} \frac{1-(1-O(x^2))}{x+O(x^3)}$

$= \lim_{x\to 0^-} \frac{O(x^2)}{x+O(x^3)} = \lim_{x\to 0^-} \frac{O(x)}{1+O(x^2)}$

$= \frac{\lim_{x\to 0^-} O(x)}{\lim_{x\to 0^-} 1+O(x^2)} = \frac{0}{1+0} = 0$

2(vii) $\lim_{x\to 0} \frac{e^x - \sin x}{\cos x} = \lim_{x\to 0} \frac{1+O(x^2)-O(x)}{1+O(x^3)}$

$= \frac{\lim_{x\to 0} 1+O(x)}{\lim_{x\to 0} 1+O(x^3)} = \frac{1}{1} = 1$

2(viii) $\lim_{x\to 0^+} \frac{x e^{-x^2} - \sin x}{2x e^x + \ln(1-2x)}$

$x e^{-x^2} = x(1-x^2+O(x^4)) = \cancel{x} - x^3 + O(x^5)$ as $x \to 0$

$\sin x = \cancel{x} - \frac{x^3}{6} + O(x^5)$

$2xe^x = 2x\left(1+x+\frac{x^2}{2}+O(x^3)\right) = \cancel{2x} + \cancel{2x^2} + x^3 + O(x^4)$

$\ln(1-2x) = -\cancel{2x} - \frac{\cancel{(2x)^2}}{\cancel{2}} - \frac{(2x)^3}{3} + O(x^4)$

2(viii) continues...

$$\lim_{x\to 0^+} \frac{x e^{-x^2} - \sin x}{2x e^x + \ln(1-2x)}$$

$$= \lim_{x\to 0^+} \frac{-x^3 + \frac{x^3}{6} + O(x^5)}{x^3 - \frac{8}{3}x^3 + O(x^4)}$$

$$= \lim_{x\to 0^+} \frac{-\frac{5}{6} + O(x^2)}{-\frac{5}{3} + O(x)}$$

$$= \frac{\lim_{x\to 0^+} -\frac{5}{6} + O(x^2)}{\lim_{x\to 0^+} -\frac{5}{3} + O(x)} = \frac{-\frac{5}{6}}{-\frac{5}{3}} = \frac{3}{6} = \frac{1}{2}$$

2(ix) $\lim_{x\to 0} \frac{1-\sqrt{1+4x^2}}{\sin^4 x}$

$$\sqrt{1+u} = 1 + \binom{1/2}{1}u + \binom{1/2}{2}u^2 + O(u^3) \quad \text{as } u\to 0$$

$$= 1 + \tfrac{1}{2}u - \tfrac{1}{8}u^2 + O(u^3)$$

$$\sqrt{1+4x^2} = 1 + \tfrac{1}{2}\cdot 4x^2 - \tfrac{1}{8}\cdot 16x^4 + O(x^6) \quad \text{as } x\to 0$$

$$\sin^4(x) = (x - O(x^3))^4 = x^4 + O(x^6)$$

2(ix)

$$\lim_{x\to 0} \frac{1-\sqrt{1+4x^2}}{\sin^4 x} = \lim_{x\to 0} \frac{-2x^2-2x^4+O(x^6)}{x^4+O(x^6)}$$

$$= \lim_{x\to 0} \frac{-\frac{2}{x^2}-2+O(x^2)}{1+O(x^2)}$$

$$= \frac{\lim_{x\to 0} -\frac{2}{x^2}-2+O(x^2)}{\lim_{x\to 0} 1+O(x^2)}$$

$$= \frac{-\infty-2+0}{1} = -\infty$$

2(x) $\lim_{x\to 0} \frac{x+(x-1)\ln(x+1)}{xe^x-\sin x}$

$\ln(1-u) = -u-\frac{u^2}{2}-\frac{u^3}{3}+O(u^4)$ as $u\to 0$

$-\ln(x+1) = -\cancel{x}+\frac{x^2}{2}-\frac{x^3}{3}+O(x^4)$ as $x\to 0$

$x\ln(x+1) = x^2-\frac{x^3}{2}+O(x^4)$

Therefore

$\cancel{x}+(x-1)\ln(x+1) = \frac{3}{2}x^2+O(x^3)$ as $x\to 0$

$\lambda(x)$ continues...

$$xe^x = x(1 + x + O(x^2)) = \cancel{x} + x^2 + O(x^3)$$

$$\sin x = \cancel{x} - \frac{x^3}{6} + O(x^5)$$

Therefore

$$xe^x - \sin x = x^2 + O(x^3)$$

It follows that

$$\lim_{x \to 0} \frac{x + (x-1)\ln(x+1)}{xe^x - \sin x} = \lim_{x \to 0} \frac{\frac{3}{2}x^2 + O(x^3)}{x^2 + O(x^3)}$$

$$= \lim_{x \to 0} \frac{\frac{3}{2} + O(x)}{1 + O(x)}$$

$$= \frac{\lim_{x \to 0} \frac{3}{2} + O(x)}{\lim_{x \to 0} 1 + O(x)} = \frac{3}{2}$$