~ Key ~

Honors Math 182 Exam 1 Version B

1. Solve the following indefinite integrals:

(i) $\int (x^7 + 7^x)\,dx$

$$= \frac{x^8}{8} + \frac{7^x}{\ln 7}$$

(ii) $\int \frac{1}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}}\,dx$

$u = \frac{x}{2},\quad dx = 2\,du$

$$= \int \frac{1}{\sqrt{1+u^2}}\,du$$

$u = \sinh t \quad du = \cosh t\,dt$

$$= \int \frac{\cosh t\,dt}{\sqrt{1+\sinh^2 t}} = \int dt = t = \operatorname{asinh} u = \operatorname{asinh}\left(\frac{x}{2}\right)$$

(iii) $\int x \ln(3x)\,dx$

$u = \ln 3x \qquad du = \frac{1}{x}\,dx$

$dv = x\,dx \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\ln 3x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln 3x - \frac{x^2}{4}$$

(iv) $\int x\sqrt{x+8}\,dx = \int (u-8)\sqrt{u}\,du = \int (u^{3/2} - 8u^{1/2})\,du$

$u = x+8 \quad du = dx \quad x = u-8$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}\cdot 8u^{3/2} = \frac{2}{5}(x+8)^{5/2} - \frac{16}{3}(x+8)^{3/2}$$

2. Solve the following definite integrals:

(i) $\int_0^2 |x-1|\,dx \quad = \int_0^1 -(x-1)\,dx + \int_1^2 (x-1)\,dx$

$= 2\int_0^1 (1-x)\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_0^1 = 2\left(1-\frac{1}{2}\right) = 1$

(ii) $\int_0^1 \frac{e^x}{4+e^{2x}}\,dx \quad = \frac{1}{4}\int_{1/2}^{e/2} \frac{2}{1+u^2}\,du$

$u = \frac{e^x}{2} \quad du = \frac{e^x}{2}\,dx$

$= \frac{1}{2}\arctan u \Big|_{1/2}^{e/2} = \frac{1}{2}\arctan\frac{e}{2} - \frac{1}{2}\arctan 1/2$

$= \frac{1}{2}\left(\arctan\frac{e}{2} \quad - \arctan\frac{1}{2}\right)$

(iii) $\int_0^2 \frac{1}{x^2+3x+2}\,dx \quad = \int_0^2 \frac{1}{(x+1)(x+2)}\,dx = \int_0^2 \left(\frac{A}{x+1} + \frac{B}{x+2}\right)dx$

$= A\ln(x+1) + B\ln(x+2)\Big|_0^2 = A\ln 3 + B\ln 4 - B\ln 2$

$A(x+2) + B(x+1) = 1$

$(A+B)x + 2A + B = 1$

$A+B=0 \quad 2A+B=1$

$B=-A \quad A=1 \quad B=-1$

$= \ln 3 - \ln 4 + \ln 2$

$= \ln 3 - 2\ln 2 + \ln 2$

$= \ln 3 - \ln 2.$

(iv) $\int_0^{\pi/6} (\sin 2x)(\cos x)\,dx \quad = \quad 2\int_0^{\pi/6} \sin x \cos^2 x\,dx$

$u = \cos x \quad du = -\sin x\,dx$

$= -2\int_1^{\sqrt{3}/2} u^2\,du = 2\int_{\sqrt{3}/2}^1 u^2\,du = \frac{2}{3}u^3\Big|_{\sqrt{3}/2}^1$

$= \frac{2}{3}\left(1 - \frac{3\sqrt{3}}{8}\right)$

3. Find the following derivatives:

(i) $\dfrac{d}{dx}\ln(1+\cos^2 x) = \dfrac{-1}{1+\cos^2 x} \cdot 2\cos x \cdot \sin x$

(ii) $\dfrac{d}{dx}\ln\sqrt{\dfrac{9+x^2}{9-x^2}} = \dfrac{d}{dx}\dfrac{1}{2}\left(\ln(9+x^2) - \ln(9-x^2)\right)$

$= \dfrac{1}{2}\left(\dfrac{2x}{9+x^2} + \dfrac{2x}{9-x^2}\right) = \dfrac{x}{9+x^2} + \dfrac{x}{9-x^2}$ for $|x| < 3$.

(iii) $\dfrac{d}{dx}|\arctan x|^3 = 3|\arctan x|^2 \dfrac{\arctan x}{|\arctan x|} \cdot \dfrac{1}{1+x^2}$

$= \dfrac{3|\arctan x|\arctan x}{1+x^2}$

(iv) $\dfrac{d}{dx}\dfrac{\cosh 2x}{x} = \dfrac{(2\sinh 2x)x - \cosh 2x}{x^2}$

4. State and prove the integration by parts formula for definite integrals.

Suppose $w(x)=f(x)g(x)$ where f and g are differentiable on $[a,b]$. By the fundamental theorem of Calculus followed by the chain rule, we have that

$$w(b)-w(a)=\int_a^b w'(x)\,dx=\int_a^b f'(x)g(x)\,dx+\int_a^b f(x)g'(x)\,dx$$

Therefore

$$\int_a^b f(x)g'(x)\,dx=f(x)g(x)\Big|_a^b-\int_a^b f'(x)g(x)\,dx.$$

5. Make the substitution $u = \ln x$ in the following integrals, but DO NOT SOLVE THEM!

(i) $\displaystyle\int_1^e x\,dx$

$u=\ln x \qquad x=e^u \qquad dx=e^u\,du$

$$=\int_0^1 e^u\cdot e^u\,du=\int_0^1 e^{2u}\,du$$

(ii) $\displaystyle\int_2^4 \frac{\sin x}{\ln x}\,dx \;=\; \int_{\ln 2}^{\ln 4}\frac{\sin e^u}{u}\cdot e^u\,du$

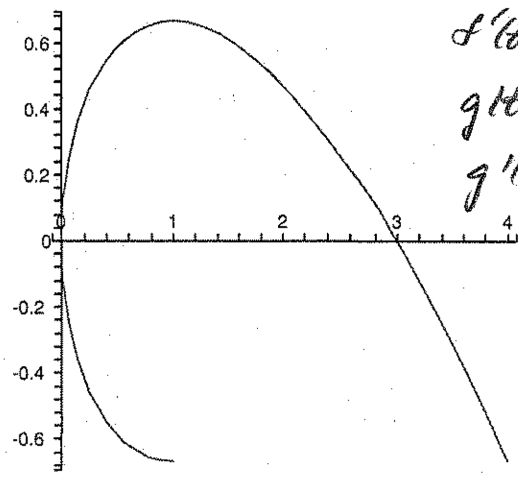
6. Find the length of the curve

$f(t) = t^2$

$f'(t) = 2t$

$g(t) = t - \frac{1}{3}t^3$

$g'(t) = 1 - t^2$



given by $(f(t), g(t))$ where t ranges over $[-1, 2]$ and $f(t) = t^2$ and $g(t) = t - \frac{1}{3}t^3$.

$$L = \int_{-1}^{2} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$= \int_{-1}^{2} \sqrt{4t^2 + (1-t^2)^2}\, dt = \int_{-1}^{2} \sqrt{4t^2 + 1 - 2t^2 + t^4}\, dt$$

$$= \int_{-1}^{2} \sqrt{1 + 2t^2 + t^4}\, dt = \int_{-1}^{2} \sqrt{(1+t^2)^2}\, dt$$

$$= \int_{-1}^{2} (1+t^2)\, dt = t + \frac{t^3}{3} \Big|_{-1}^{2} =$$

$$= 2 + \frac{8}{3} + 1 + \frac{1}{3} = 3 + \frac{9}{3} = 3 + 3 = 6.$$