Honors Math 182 Exam 1 Version A

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1. Solve the following indefinite integrals:

(i) $\int (x^3 + 3^x)\,dx$ $= \frac{x^4}{4} + \int e^{x\ln 3}\,dx$

$\dots = \frac{x^4}{4} + \frac{1}{\ln 3} e^{x\ln 3} = \boxed{\frac{x^4}{4} + \frac{3^x}{\ln 3}}$

(ii) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ $= \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$

$u = x/2 \quad x = 2u \quad dx = 2du$

$\dots = \arcsin u = \boxed{\arcsin\left(\frac{x}{2}\right)}$

(iii) $\int \arctan\sqrt{x}\,dx$ $= 2\int u \arctan u\,du$

$u = \sqrt{x},\ x = u^2,\ dx = 2u\,du \qquad p = \arctan u \qquad dp = \frac{1}{1+u^2}\,du$

$dq = u\,du \qquad q = \frac{1}{2}u^2$

$= u^2\arctan u + \int \frac{u^2}{1+u^2}\,du = u^2\arctan u + \int \left(1 - \frac{1}{1+u^2}\right)du$

$= u^2 \arctan u + u - \arctan u = x\arctan\sqrt{x} + \sqrt{x} - \arctan\sqrt{x}$

$= \boxed{(x-1)\arctan\sqrt{x} + \sqrt{x}}$

(iv) $\int x^3 e^{2x^2}\,dx$ $= \frac{1}{8}\int 2x^2 e^{2x^2} \cdot 4x\,dx = \frac{1}{8}\int u e^u\,du$

$u = 2x^2 \quad du = 4x\,dx \qquad p = u \qquad dp = du$

$dq = e^u\,du \qquad q = e^u$

$= \frac{1}{8}\left(ue^u - \int e^u\,du\right) = \frac{1}{8}(ue^u - e^u) = \frac{1}{8}(u-1)e^u$

$= \boxed{\frac{1}{8}(2x^2-1)e^{2x^2}}$

2. Solve the following definite integrals:

(i) $\int_0^1 \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{e^u}\,du = \frac{1}{2}\int_0^1 e^{-u}\,du$

$u = x^2 \quad du = 2x\,dx$

$= -\frac{1}{2}e^{-u}\Big|_0^1 = -\frac{1}{2}e^{-1} + \frac{1}{2} = \boxed{\frac{1}{2}\left(1 - \frac{1}{e}\right)}.$

(ii) $\int_{-4}^1 x\sqrt{x+8}\,dx = \int_4^9 (u-8)\sqrt{u}\,du = \int_4^9 (u^{3/2} - 8u^{1/2})\,du$

$u = x+8, \; x = u-8, \; dx = du$

$= \frac{2}{5}u^{5/2} - \frac{16}{3}u^{3/2}\Big|_9^4 = \frac{2}{5}(2^5 - 3^5) - \frac{16}{3}(2^3 - 3^3)$

$= \frac{2}{5}(32 - 243) - \frac{16}{3}(8 - 27) = -\frac{422}{5} + \frac{304}{3} = \boxed{\frac{-254}{15}}$

$81 \times 3 = 243$

$243 - 32 = 211$

$211 \times 6 = 1266$

$27 - 8 = 19$

$19 \times 16 = 304$

$304 \times 5 = 1520$

$1520 - 1266 = 254$

(iii) $\int_0^1 \frac{1}{1+e^x}\,dx = \int_1^e \frac{1}{1+u}\cdot\frac{1}{u}\,du = \int_1^e \frac{A}{1+u} + \frac{B}{u}\,du$

$u = e^x, \; du = e^x dx = u\,dx$

$= A\ln(1+u) + B\ln(u)\Big|_1^e = A\left(\ln(1+e) - \ln 2\right) + B$

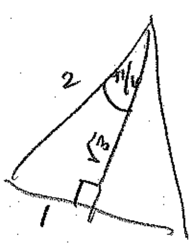
$\frac{1}{(1+u)u} = \frac{A}{1+u} + \frac{B}{u} = \frac{Au + B(1+u)}{(1+u)u} = \frac{(A+B)u + B}{(1+u)u} \quad \begin{cases} A+B=0 \\ B=1 \\ A=-1 \end{cases}$

$\cdots = \boxed{\ln 2 - \ln(1+e) + 1}$

(iv) $\int_0^{\pi/6} (\sin 2x)(\cos x)\,dx$

$= \int_0^{\pi/6} 2\sin x \cos^2 x\,dx = -2\int_1^{\sqrt{3}/2} u^2\,du = -2\,\frac{u^3}{3}\Big|_1^{\sqrt{3}/2}$

$u = \cos x, \; du = -\sin x\,dx$



$= -\frac{2}{3}\left(\left(\frac{\sqrt{3}}{2}\right)^3 - 1\right) = -\frac{2}{3}\cdot\frac{3\sqrt{3}}{8} + \frac{2}{3} = \boxed{\frac{2}{3} - \frac{\sqrt{3}}{4}}$

3. Find the following derivatives:

(i) $\dfrac{d}{dx}\ln(1+\cos^2 x)$ $= \dfrac{1}{1+\cos^2 x}\cdot 2\cos x\cdot(-\sin x)$

$= \boxed{-\dfrac{2\cos x\sin x}{1+\cos^2 x}}$

(ii) $\dfrac{d}{dx}\ln\sqrt{\dfrac{4+x^2}{4-x^2}}$ $= \dfrac{d}{dx}\dfrac{1}{2}\left(\ln(4+x^2)-\ln(4-x^2)\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{4+x^2}\cdot 2x - \dfrac{1}{4-x^2}\cdot(-2x)\right)$

$= \dfrac{x}{4+x^2} - \dfrac{x}{4-x^2} = \boxed{\dfrac{2x^3}{x^4-16} \quad \text{for } |x|<2}$

(iii) $\dfrac{d}{dx}|\arctan x|^3$ $= 3|\arctan x|^2\cdot\dfrac{\arctan x}{|\arctan x|}\cdot\dfrac{1}{1+x^2}$

$= \boxed{\dfrac{3|\arctan x|\cdot\arctan x}{1+x^2}}$

(iv) $\dfrac{d}{dx}\dfrac{\sinh 2x}{x}$ $= \dfrac{(\cosh 2x)x-(\sinh 2x)2}{x^2}$

$= \boxed{\dfrac{x\cosh 2x - 2\sinh 2x}{x^2}}$

4. State and prove the integration by parts formula for definite integrals.

Let $w(x) = f(x)g(x)$ where f and g are differentiable functions on the interval $[a,b]$. By the product rule it follows that $w'(x) = f'(x)g(x) + f(x)g'(x)$. Then by the fundamental theorem of calculus part 2

$$w(b) - w(a) = \int_a^b w'(x)\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx.$$

It follows that

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

5. Make the substitution $u = \sqrt{x}$ in the following integrals, but DO NOT SOLVE THEM!

(i) $\displaystyle\int_0^4 x\,dx$

$u = \sqrt{x} \quad x = u^2 \quad dx = 2u\,du$

$$= \int_0^2 u^2 \cdot 2u\,du = \boxed{2\int_0^2 u^3\,du}$$

(ii) $\displaystyle\int_0^2 x \operatorname{atanh}\sqrt{x}\,dx$

$u = \sqrt{x}, \quad x = u^2, \quad dx = 2u\,du$

$$= \int_0^{\sqrt{2}} u^2 \operatorname{atanh} u \cdot 2u\,du = \boxed{2\int_0^{\sqrt{2}} u^3 \operatorname{atanh} u\,du}$$

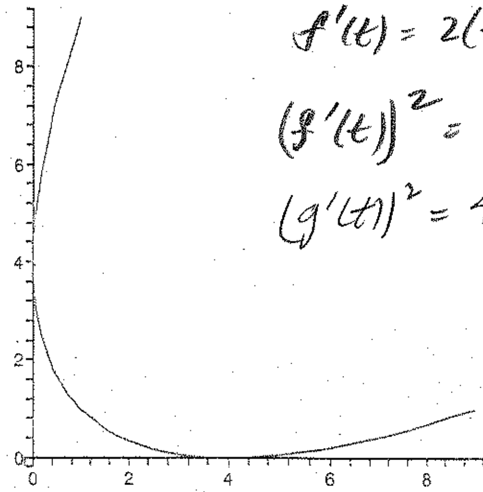
6. Find the length of the curve

$f(t) = (t-1)^2 \quad g(t) = (t+1)^2$

$f'(t) = 2(t-1) \quad g'(t) = 2(t+1)$

$(f'(t))^2 = 4(t-1)^2 = 4(t^2-2t+1)$

$(g'(t))^2 = 4(t+1)^2 = 4(t^2+2t+1)$



given by $(f(t), g(t))$ where t ranges over $[-2, 2]$ and $f(t) = (t-1)^2$ and $g(t) = (t+1)^2$.

$$L = \int_{-2}^{2} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = 2\int_0^2 \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$= 4\int_0^2 \sqrt{(t^2-2t+1) + (t^2+2t+1)}\, dt = 4\sqrt{2}\int_0^2 \sqrt{t^2+1}\, dt$$

Now work the integral... $\cosh^2 u - \sinh^2 u = 1$

$t = \sinh u \quad dt = \cosh u\, du$

$$= 4\sqrt{2}\int_0^{\operatorname{asinh} 2} \cosh^2 u\, du = \sqrt{2}\int_0^{\operatorname{asinh} 2} (e^u + e^{-u})^2 du = \sqrt{2}\int_0^{\operatorname{asinh} 2} (e^{2u} + 2 + e^{-2u})\, du$$

$$= \sqrt{2}\left(\tfrac{1}{2}e^{2u} + 2u - \tfrac{1}{2}e^{-2u}\right)\Big|_0^{\operatorname{asinh} 2} = 2\sqrt{2}\left(\sinh u \cosh u + u\right)\Big|_0^{\operatorname{asinh} 2}$$

$$= 2\sqrt{2}\left(2\sqrt{2^2+1} + \operatorname{asinh} 2\right)$$

$$= \boxed{4\sqrt{10} + 2\sqrt{2}\,\operatorname{asinh} 2}$$