Honors Math 182 Exam 2 Version B

Key

1. Find to 5 digit accuracy the following definite integrals:

(i) $\displaystyle\int_1^2 (1+u)^{1/3}\,du$

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I1:= Int((1+u)^(1/3),u=1..2);
evalf(I1);
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≈ 1.35518

(ii) $\displaystyle\int_0^{\pi} \frac{\cos x + \sin^2 x}{\sin x + \cos^2 x}\,dx$

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I2:= Int((cos(x)+sin(x)^2)/
        (sin(x)+cos(x)^2),x=0..Pi);
evalf(I2);
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≈ 1.43773

(iii) $\displaystyle\int_0^1 \frac{1}{\sqrt{1+y+y^2+y^3}}\,dy$

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I3:=Int(1/sqrt(1+y+y^2+y^3), y=0..1);
evalf(I3);
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≈ 0.73754

(iv) $\displaystyle\int_1^e (\ln t)^8\,dt$

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I4:= Int(ln(t)^8, t=1..exp(1));
evalf(I4);
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≈ 0.27436

2. The Taylor's formula for $\cos x$ when $a = 0$ is

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} + R_n(x) \qquad \text{where} \qquad R_n(x) = \frac{(-1)^{n+1}}{(2n+2)!} x^{2n+2} \cos \xi$$

and ξ is some number between 0 and x. Use the inequality $|\cos \xi| \le 1$ to

(i) Show that $R_n(5) \to 0$ as $n \to \infty$.

$$|R_n(5)| = \frac{5^{2n+2}}{(2n+2)!} |\cos \xi| \le \frac{5^{2n+2}}{(2n+2)!}$$

$$= \frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot \cdots 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots 2n+2} \le \frac{5^5}{5!} \left(\frac{5}{6}\right)^{2n-3} \to 0$$

as $n \to \infty$

(ii) Estimate how large n has to be in order to guarantee $|R_n(5)| \le 0.5 \times 10^{-4}$.

n	$\frac{5^{2n+2}}{(2n+2)!}$
8	0.000596
9	0.000037 < 0.00005

so $n = 9$

(iii) Show that $R_2(x) = \mathcal{O}(x^6)$ as $x \to 0$.

$$\left|\frac{R_2(x)}{x^6}\right| \le \frac{|x|^6 \, |\cos \xi|}{6! \, |x^6|} \le \frac{1}{6!}$$

so $M = \frac{1}{6!}$ is a bound for $\left|\frac{R_2(x)}{x^6}\right|$ for all x.

Consequently $R_2(x) = \mathcal{O}(x^6)$

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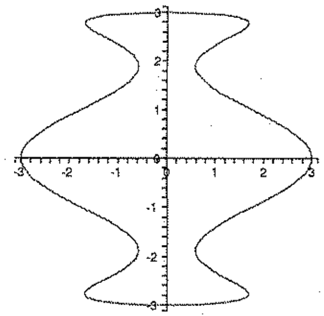
3. Consider the closed curve $(f(t), g(t))$ where $0 \le t \le 2\pi$ given by

$$f(t) = 2\cos t + \cos 5t \qquad \text{and} \qquad g(t) = 3\sin t.$$

(i) Find to 5 digit accuracy the length of this curve.



$f'(t) = -2\sin t - 5\sin 5t$

$g'(t) = 3\cos t$

$L = \int_0^{2\pi} \sqrt{f'(t)^2 + g'(t)^2}\, dt \approx 25.5156$

(ii) Find to 5 digit accuracy the area enclosed by the curve.

$A = \int_0^{2\pi} f(t)\, g'(t)\, dt \approx 18.8496$

(iii) Find the equation of the line tangent to the curve at the point $(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$.

$3\sin t = \frac{3\sqrt{2}}{2}$, $\sin t = \frac{\sqrt{2}}{2}$ so $t = \pi/4$

$m = \dfrac{g'(\pi/4)}{f'(\pi/4)} = 1$

Equation of line $y - y_0 = m(x - x_0)$

$y - \frac{3\sqrt{2}}{2} = x - \frac{\sqrt{2}}{2}$ $\qquad y = x + \sqrt{2}$

(iv) Find the radius of curvature ρ of the curve at the point $(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$.

$f''(t) = -2\cos t - 25\cos 5t$, $g''(t) = -3\sin t$

$K = \left.\dfrac{g''(t)f'(t) - g'(t)f''(t)}{(f'(t)^2 + g'(t)^2)^{3/2}}\right|_{t=\pi/4} = \dfrac{-13}{9}$

$\rho = \dfrac{1}{|K|} = \dfrac{9}{13}$

4. Suppose $f(x) = \mathcal{O}(x^5)$ and $g(x) = \mathcal{O}(x)$ as $x \to 0$.

(i) Show $f(x) + g(x) = \mathcal{O}(x)$ as $x \to 0$.

Let $\delta_1 > 0$ and M_1 be chosen so $|x| < \delta_1$ implies $\left|\frac{f(x)}{x^5}\right| \le M_1$
Let $\delta_2 > 0$ and M_2 be chosen so $|x| < \delta_2$ implies $\left|\frac{g(x)}{x}\right| \le M_2$.

Choose $\delta = \min(\delta_1, \delta_2)$ and $M = M_1 \delta^4 + M_2$.
Then $|x| < \delta$ implies $|x| < \delta_1$ and $|x| < \delta_2$.
Thus

$$\left|\frac{f(x) + g(x)}{x}\right| \le \left|\frac{f(x)}{x^5}\right| |x^4| + \left|\frac{g(x)}{x}\right| \le M_1 \delta^4 + M_2 = M$$

Therefore $f(x) + g(x) = O(x)$ as $x \to 0$.

(ii) Show $f(x)g(x) = \mathcal{O}(x^6)$ as $x \to 0$.

Let $\delta_1 > 0$ and M_1 be chosen so $|x| < \delta_1$ implies $\left|\frac{f(x)}{x^5}\right| \le M_1$
Let $\delta_2 > 0$ and M_2 be chosen so $|x| < \delta_2$ implies $\left|\frac{g(x)}{x}\right| \le M_2$

Choose $\delta = \min(\delta_1, \delta_2)$ and $M = M_1 M_2$.
Then $|x| < \delta$ implies $|x| < \delta_1$ and $|x| < \delta_2$.
Thus

$$\left|\frac{f(x) g(x)}{x^6}\right| = \left|\frac{f(x)}{x^5}\right| \left|\frac{g(x)}{x}\right| \le M_1 M_2 = M,$$

Therefore $f(x)g(x) = O(x^6)$ as $x \to 0$.

5. Find the first 3 non-zero terms of the Taylor series for $\ln(1 + x^2)$ where $a = 0$.

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + O(x^4) \quad \text{as } x \to 0$$

Therefore

$$\ln(1+x^2) = -(-x^2) - \frac{(-x^2)^2}{2} - \frac{(-x^2)^3}{3} + O(x^8)$$

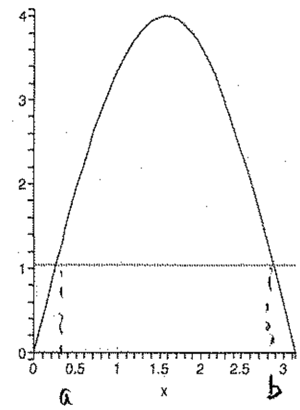
$$= \underbrace{x^2 - \frac{x^4}{2} + \frac{x^6}{3}}_{\text{These are the first 3 non-zero terms.}} + O(x^8)$$

6. Consider the region enclosed by the curve $f(x) = 4\sin x$ and $g(x) = \sqrt{6} - \sqrt{2}$.

(i) Find the volume formed by rotating this region about the x-axis.

$$a = \arcsin\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = \frac{\pi}{12}$$

$$b = \pi - \frac{\pi}{12} = \frac{11\pi}{12}$$



$$V_x = \int_a^b \pi f(x)^2\,dx - \int_a^b \pi g(x)^2\,dx$$

$$= 4\pi + \frac{10\sqrt{3}}{3}\pi^2 \approx 69.5486$$

(ii) Find the volume formed by rotating this region about the y-axis.

$$V_y = \int_a^b 2\pi x f(x)\,dx - \int_a^b 2\pi x g(x)\,dx$$

$$= \frac{\sqrt{2}}{6}\pi^2\left(5\pi - 5\sqrt{3}\pi + 12\sqrt{3} + 12\right)$$

$$= \frac{\sqrt{2}}{6}\pi^2\left(5\pi(1-\sqrt{3}) + 12(1+\sqrt{3})\right)$$

$$\approx 49.5164$$

7. Compute the limit $\lim_{x\to 0} \dfrac{\sin x - x\cos x}{x^3}$ $= \frac{1}{3}$ since

$$\frac{\sin x - x\cos x}{x^3} = \frac{\cancel{x} - \frac{x^3}{3!} + O(x^5) - x\left(\cancel{1} - \frac{x^2}{2} + O(x^4)\right)}{x^3}$$

$$= \frac{x^3\left(\frac{1}{2} - \frac{1}{6}\right) + O(x^5)}{x^3} = \frac{1}{3} + O(x^2) \to \frac{1}{3}$$

as $x \to 0$