Honors Math 182 Homework 1 Version A

1. Find the following derivatives. Work the problems using pencil and paper. You may check your work with Maple.

(i) $\dfrac{d}{dx}\left(x^\pi + \pi^x\right)$

(ii) $\dfrac{d}{dx}\left(x^2 \ln(x^2+1)\right)$

(iii) $\dfrac{d}{dx} \arctan(5 \sin x)$

(iv) $\dfrac{d}{dx}|2+x|^{\sin x}$

(v) $\dfrac{d}{dx}\dfrac{\tan(e^x)}{1+x^4}$

2. Find the partial fraction decompositions of the following rational expressions.

(i) $\dfrac{3x}{(x+1)(x-2)}$

(ii) $\dfrac{x^2}{(x^2+1)(x-2)}$

(iii) $\dfrac{x+3}{(x+1)^2(x-2)}$

3. Work the following story problems from the handouts.

(i) Stewart: page 311 # 12, 13, 14, 22, 30

(ii) Ellis and Gulick: page 197 # 7, 9, 15, 17, 23

From Single Variable Calculus — James Stewart

4.1 Exercises

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.
   (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

4. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

5. If $y = x^3 + 2x$ and $dx/dt = 5$, find dy/dt when $x = 2$.

6. If $x^2 + y^2 = 25$ and $dy/dt = 6$, find dx/dt when $y = 4$.

7. If $z^2 = x^2 + y^2$, $dx/dt = 2$, and $dy/dt = 3$, find dz/dt when $x = 5$ and $y = 12$.

8. A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point (2, 3), the y-coordinate is increasing at a rate of 4 cm/s. How fast is the x-coordinate of the point changing at that instant?

9–12
(a) What quantities are given in the problem?
(b) What is the unknown?
(c) Draw a picture of the situation for any time t.
(d) Write an equation that relates the quantities.
(e) Finish solving the problem.

9. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

10. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 P.M.?

11. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

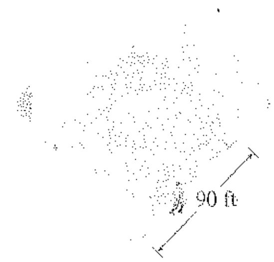
12. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

13. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

14. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

15. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

16. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
   (a) At what rate is his distance from second base decreasing when he is halfway to first base?
   (b) At what rate is his distance from third base increasing at the same moment?



17. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

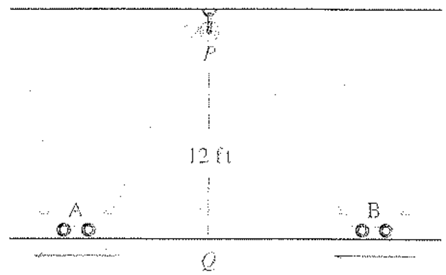
18. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

19. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 P.M.?

20. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point (4, 2), its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

21. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure on the next page). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of

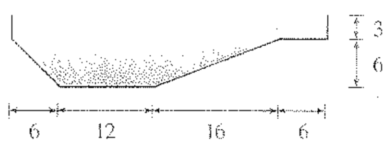
2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



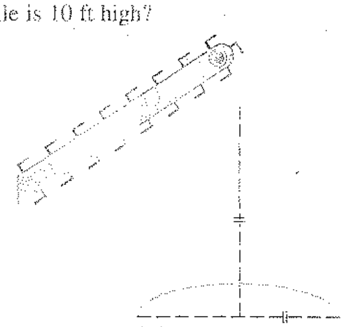
22. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

23. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?

24. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft^3/min, how fast is the water level rising when the depth at the deepest point is 5 ft?



25. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



26. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?

27. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

28. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°?

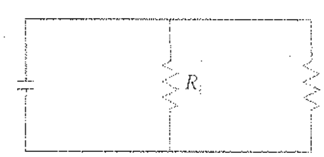
29. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

30. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

31. If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of 0.3 Ω/s and 0.2 Ω/s, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



32. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?

33. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.
(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

34. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

35. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

36. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

37. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

38. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

4.2 Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maximum and minimum values.

1 Definition A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Figure 1 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

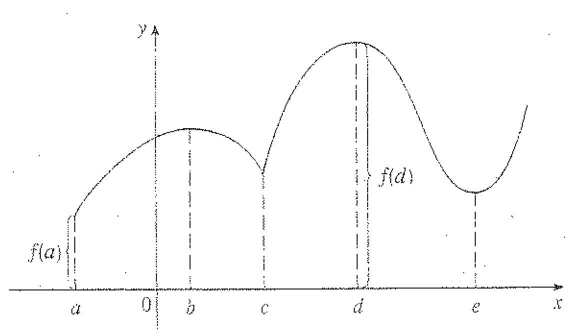


FIGURE 1
Minimum value $f(a)$, maximum value $f(d)$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all.

4.6 Exercises

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
.	.	.
.	.	.

   (b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

7. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
   (a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the total area.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the total area as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

8. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
   (a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the volume.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the volume as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

9. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

10. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

11. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
    (b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

12. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

13. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

14. Find, correct to two decimal places, the coordinates of the point on the curve $y = \tan x$ that is closest to the point $(1, 1)$.

15. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

16. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

17. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

18. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

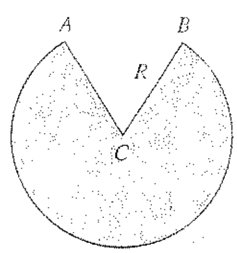
19. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus, the diameter of the semicircle is equal to the width of the rectangle. See Exercise 56 on page 24.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

20. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

21. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

22. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

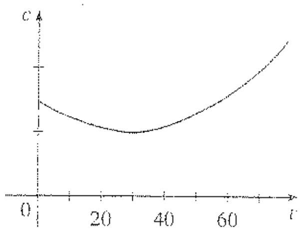
23. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



24. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

25. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

26. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



27. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

28. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.
(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

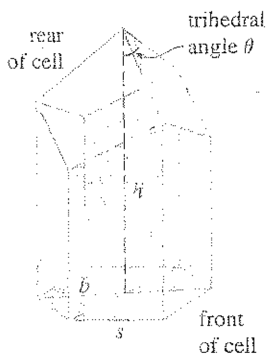
Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

29. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end as in the figure. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot\theta + \left(3s^2\sqrt{3}/2\right)\csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.
(a) Calculate $dS/d\theta$.
(b) What angle should the bees prefer?
(c) Determine the minimum surface area of the cell (in terms of s and h).

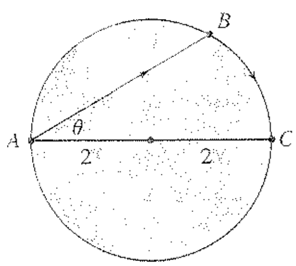
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



30. A boat leaves a dock at 2:00 P.M. and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 P.M. At what time were the two boats closest together?

31. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

32. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?

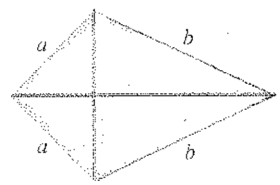


33. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

34. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

35. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

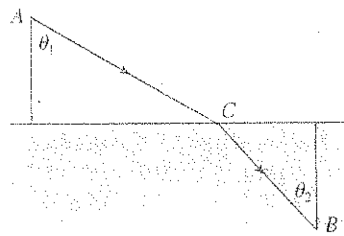
CAS **36.** The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



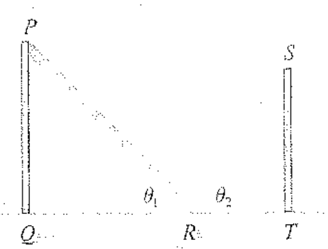
37. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

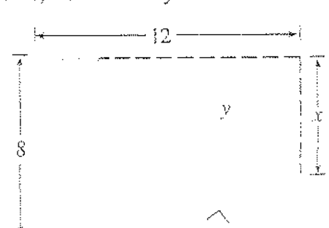
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



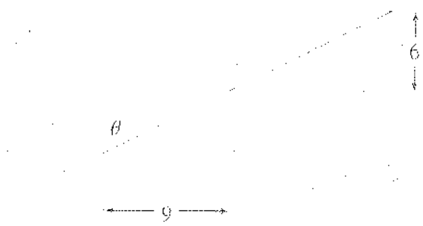
38. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



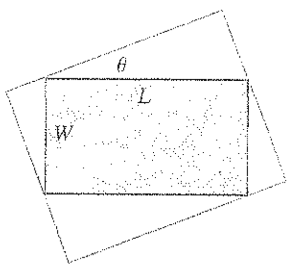
39. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



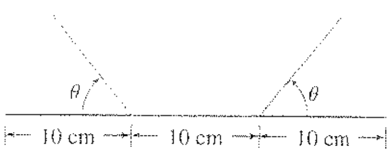
40. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



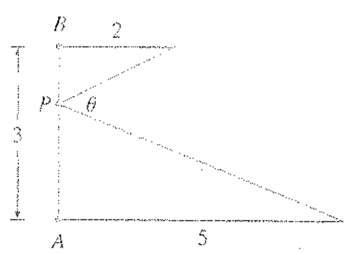
41. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W.



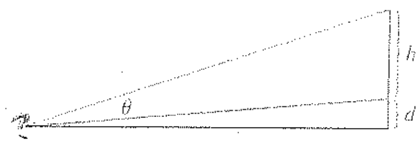
42. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



43. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



44. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



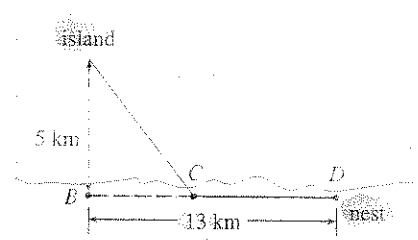
45. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

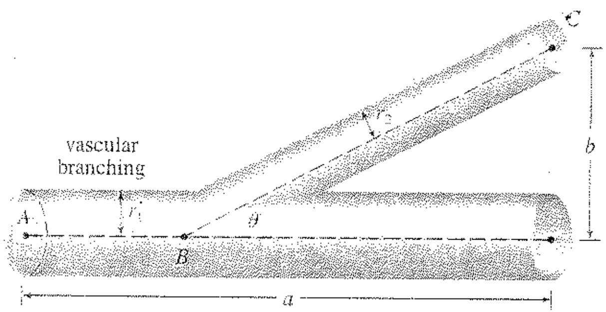
(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than land?



46. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 6.6.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

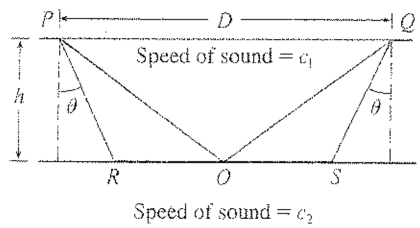
(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

47. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q.

(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.

(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.

(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, $T_3 = 0.34$ s. Find c_1, c_2, and h.



Note: Geophysicists use this technique when studying the structure of Earth's crust, whether searching for oil or examining fault lines.

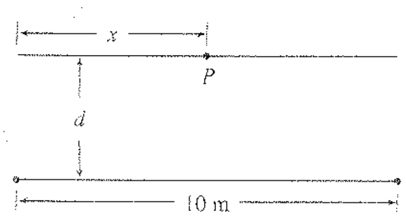
48. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.



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for the value of t_0 at which $x = 24$. By the similar triangles appearing in Figure 3.28,

$$\frac{x+y}{20} = \frac{y}{6}$$

so that

$$6x + 6y = 20y$$

or equivalently,

$$y = \frac{3}{7}x$$

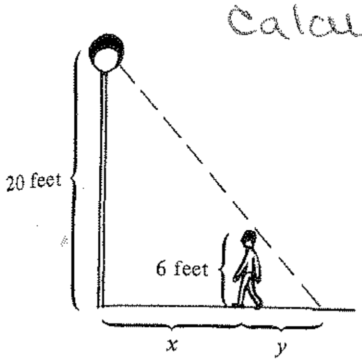


FIGURE 3.27

Therefore, differentiating with respect to t, we obtain

$$\frac{dy}{dt} = \frac{3}{7}\frac{dx}{dt}$$

By hypothesis, $dx/dt = -5$, so that when we substitute we obtain

$$\frac{dy}{dt} = \frac{3}{7}(-5) = -\frac{15}{7}$$

Consequently Pat's shadow shrinks at the rate of $\frac{15}{7}$ feet per second at the moment in question. □

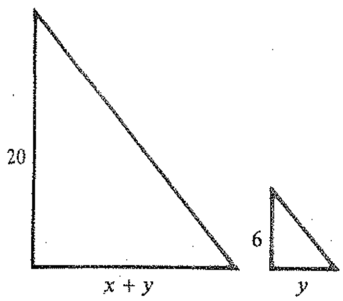


FIGURE 3.28

The procedure we have used in solving the related rates examples of this section includes the following steps:

1. Identify and label the different variables. Include the variable whose rate is to be evaluated and those whose rates are given. It may be helpful to sketch a drawing at this stage.
2. Determine an equation connecting the variables appearing in step 1.
3. Differentiate both sides of the equation implicitly, and solve for the derivative that will yield the desired rate.
4. Evaluate the derivative by using the given values of the variables and their rates.

This procedure should also help you in solving related rates problems.

EXERCISES 3.8

1. Suppose the radius of a spherical balloon is shrinking at $\frac{1}{2}$ inch per minute. How fast is the volume decreasing when the radius is 4 inches?
2. Suppose a snowball remains spherical while it melts, with the radius shrinking at one inch per hour. How fast is the volume of the snowball decreasing when the radius is 2 inches?
3. Suppose the volume of the snowball in Exercise 2 shrinks at the rate of $dV/dt = -2/V$ (cubic inches per hour). How fast is the radius changing when the radius is $\frac{1}{2}$ inch?

4. A spherical balloon is inflated at the rate of 3 cubic inches per minute. How fast is the radius of the balloon increasing when the radius is 6 inches?

5. Suppose a spherical balloon grows in such a way that after t seconds, $V = 4\sqrt{t}$ (cubic inches). How fast is the radius changing after 64 seconds?

6. A spherical balloon is losing air at the rate of 2 cubic inches per minute. How fast is the radius of the balloon shrinking when the radius is 8 inches?

7. Water leaking onto a floor creates a circular pool whose area increases at the rate of 3 square inches per minute. How fast is the radius of the pool increasing when the radius is 10 inches?

8. A point moves around the circle $x^2 + y^2 = 9$. When the point is at $(-\sqrt{3}, \sqrt{6})$, its x coordinate is increasing at the rate of 20 units per second. How fast is its y coordinate changing at that instant?

9. A ladder 15 feet long leans against a vertical wall. If the bottom end of the ladder is pulled away from the wall at a rate of 1 foot per second, at what rate does the top of the ladder slip down the wall when the bottom of the ladder is 5 feet from the wall?

10. Suppose the top of the ladder in Exercise 9 is being pushed up the wall at the rate of 1 foot per second. How fast is the base of the ladder approaching the wall when it is 3 feet from the wall?

11. A board 5 feet long slides down a wall. At the instant the bottom end is 4 feet from the wall, the other end is moving down the wall at the rate of 2 feet per second. At that moment,
    a. how fast is the bottom end sliding along the ground?
    b. how fast is the area of the region between the board, ground, and wall changing?

12. Suppose the water in Example 4 is poured in at the rate of $\frac{3}{2}$ cubic inches per second. How fast is the water level rising when the water is 2 inches deep?

13. Suppose that the water level in Example 4 is rising at $\frac{1}{2}$ inch per second. How fast is the water being poured in when the water has a depth of 2 inches?

14. Water is released from a conical tank with height 50 feet and radius 30 feet and falls into a rectangular tank whose base has an area of 400 square feet (Figure 3.29). The rate of release is controlled so that when the height of the water in the conical tank is x feet, the height is decreasing at the rate of $50 - x$ feet per minute. How fast is the water level in the rectangular tank rising when the height of the water in the conical tank is 10 feet? (*Hint:* The total amount of water in the two tanks is constant.)

15. A water trough is 12 feet long, and its cross section is an equilateral triangle with sides 2 feet long. Water is pumped into the trough at a rate of 3 cubic feet per minute. How fast is the water level rising when the depth of the water is $\frac{1}{2}$ foot?

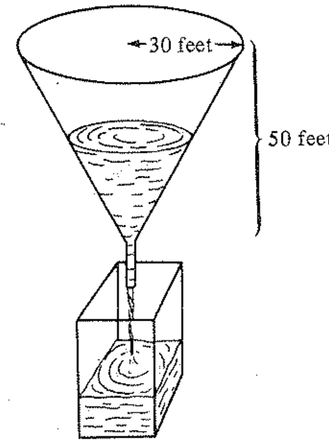


FIGURE 3.29

16. A rope is attached to the bow of a sailboat coming in for the evening. Assume that the rope is drawn in over a pulley 5 feet higher than the bow at the rate of 2 feet per second, as shown in Figure 3.30. How fast is the boat docking when the length of rope from bow to pulley is 13 feet?

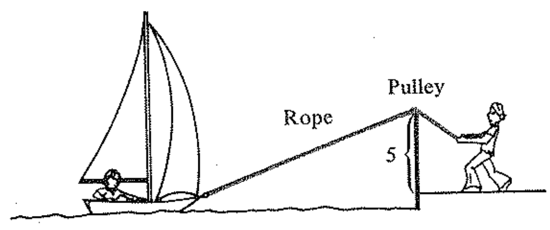


FIGURE 3.30

17. Suppose the rope in Exercise 16 is pulled so that the boat docks at a constant rate of 2 feet per second. How fast is the rope being pulled in when the boat is 12 feet from the dock?

18. As in Exercise 16, assume that the boat is pulled in by a rope attached to the bow passing through a pulley 5 feet above the bow. Assume also that the distance between the bow and the dock decreases as the cube root of the distance; that is, if the distance at time t is y feet, then $dy/dt = -y^{1/3}$ (feet per second). How fast is the length of the rope shrinking when the bow is 8 feet from the dock?

19. A spotlight is on the ground 100 feet from a building that has vertical sides. A person 6 feet tall starts at the spotlight and walks directly toward the building at a rate of 5 feet per second.
   a. How fast is the top of the person's shadow moving down the building when the person is 50 feet away from it?
   b. How fast is the top of the shadow moving when the person is 25 feet away?

20. A kite 100 feet above the ground is being blown away from the person holding its string, in a direction parallel to the ground and at the rate of 10 feet per second. At what rate must the string be let out when the length of string already let out is 200 feet?

21. When a rocket is two miles high, it is moving vertically upward at a speed of 300 miles per hour. At that instant, how fast is the angle of elevation of the rocket increasing, as seen by an observer on the ground 5 miles from the launching pad?

22. A street light 16 feet high casts a shadow on the ground from a ball that is dropped from a height of 16 feet but 15 feet from the light. How fast is the shadow moving along the ground when the ball is 5 feet from the ground. (*Note:* The distance s from the ball to the ground t seconds after release is given by the equation $s = 16 - 16t^2$.)

23. A person is pushing a box up the ramp in Figure 3.31 at the rate of 3 feet per second. How fast is the box rising?

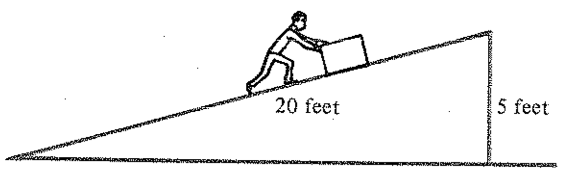


FIGURE 3.31

24. Maple and Main Streets are straight and perpendicular to each other. A stationary police car is located on Main Street $\frac{1}{4}$ mile from the intersection of the two streets. A sports car on Maple Street approaches the intersection at the rate of 40 miles per hour. How fast is the distance between the two cars decreasing when the moving car is $\frac{1}{8}$ mile from the intersection?

25. Suppose in Exercise 24 that as the sports car approaches the intersection, the distance between the sports car and the police car decreases at 25 miles per hour. How far from the intersection would the sports car be at the moment when it is traveling 40 miles per hour?

26. A helicopter flies parallel to the ground at an altitude of $\frac{1}{2}$ mile and at a speed of 2 miles per minute. If the helicopter flies along a straight line that passes directly over the White House, at what rate is the distance between the helicopter and the White House changing 1 minute after the helicopter flies over the White House?

27. A Flying Tiger is making a nose dive along a parabolic path having the equation $y = x^2 + 1$, where x and y are measured in feet. Assume that the sun is directly above the y axis, that the ground is the x axis, and that the distance from the plane to the ground is decreasing at the constant rate of 100 feet per second. How fast is the shadow of the plane moving along the ground when the plane is 2501 feet above the earth's surface? Assume that the sun's rays are vertical.

28. Boyle's Law states that if the temperature of a gas remains constant, then the pressure p and the volume V of the gas satisfy the equation $pV = c$, where c is a constant. If the volume is decreasing at the rate of 10 cubic inches per second, how fast is the pressure increasing when the pressure is 100 pounds per square inch and the volume is 20 cubic inches?

29. The tortoise and the hare are having their fabled footrace, each moving along a straight line. The tortoise, moving at a constant rate of 10 feet per minute, is 4 feet from the finish line when the hare wakes up 5001 feet from the finish line and darts off after the tortoise. Let x be the distance from the tortoise to the finish line, and suppose

$$y = 5001 - 2500\sqrt{4 - x}$$

is the distance from the hare to the finish line.
   a. How fast is the hare moving when the tortoise is 3 feet from the finish line?
   b. Who wins? By how many feet?

30. A 10-foot square sign of negligible thickness revolves about a vertical axis through its center at a rate of 10 revolutions per minute. An observer far away sees it as a rectangle of variable width. How fast is the width changing when the sign appears to be 6 feet wide and is increasing in width? (*Hint:* View the sign from above, and consider the angle it makes with a line pointing toward the observer.)

31. Suppose a deer is standing 20 feet from a highway on which a car is traveling at a constant rate of v feet per second. Let θ be the angle made by the highway and the line of sight from a passenger to the deer (Figure 3.32).

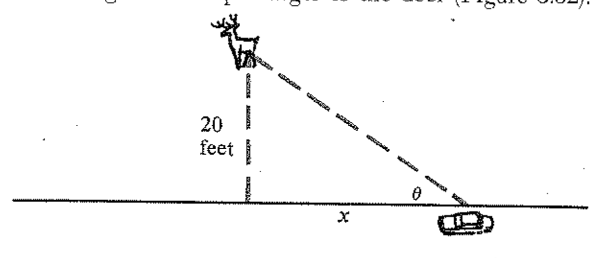


FIGURE 3.32

Show that

$$\frac{d\theta}{dt} = \frac{20v}{400 + x^2}$$

(Notice that for x close to 0, $d\theta/dt$ is approximately $v/20$, and thus for the passenger to keep the deer in focus, the passenger's eyes must rotate at the approximate rate of $v/20$ feet per second. This suggests why at large velocities it may be impossible to keep a stationary object near the highway in focus.)

°32. At night a patrol boat approaches a point on shore along the curve $y = -\frac{1}{2}x^3$, as indicated in Figure 3.33. If the boat moves along the curve so that $dx/dt = -x$, and if its spotlight is pointed straight ahead, determine how fast the illuminated spot on the shore moves when $x = -2$. (*Hint:* You will need to find the x intercept of the line tangent to the curve $y = -\frac{1}{2}x^3$.)

°33. A deer 5 feet long and 6 feet tall, whose rump is 4 feet above ground as in Figure 3.34, approaches a street light with lamp 20 feet above ground. If the deer proceeds at 3 feet per second, how fast is its shadow changing when the front of the deer is

a. 48 feet from the street light?
b. 24 feet from the street light?

(*Hint:* In parts (a) and (b), determine which yields the shadow, the head or the rump of the deer.)

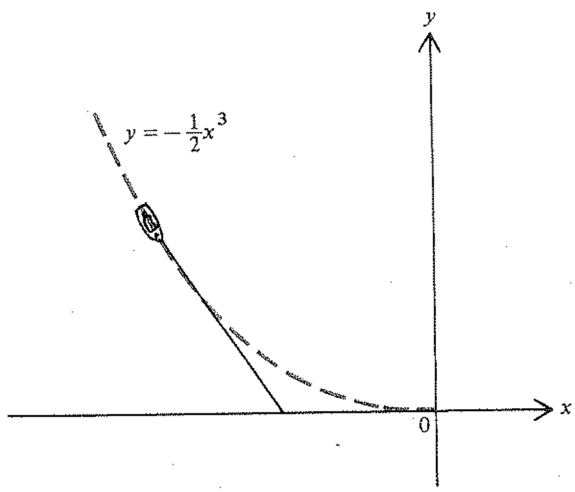


FIGURE 3.33

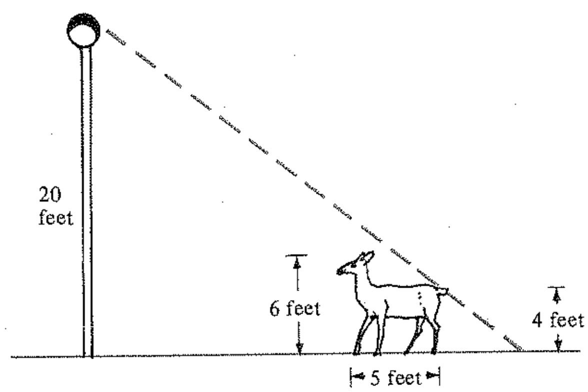


FIGURE 3.34

3.9 TANGENT LINE APPROXIMATIONS AND THE DIFFERENTIAL

As we have seen, tangent lines and derivatives are closely related. In this section we will make use of this relationship to estimate values of functions that are difficult or impossible to obtain exactly.

Tangent Line Approximations

To illustrate the problem of approximating values of functions, let

$$f(x) = \sqrt[3]{x}$$

Our idealized formula in the insulation example does not take into account all possible variables. For instance, it does not reflect installation costs or the fact that some heat is lost through the walls and floor. Nevertheless, the problem as stated and solved is representative of fairly common conditions.

There is a general procedure we have used in solving the applied problems in this section. Below we list the major features of the procedure as a guide for you in solving applied problems involving extreme values.

1. After reading the problem carefully, choose a letter for the quantity to be maximized or minimized, and choose auxiliary variables for the other quantities appearing in the problem.
2. Express the quantity to be maximized or minimized in terms of the auxiliary variables. A diagram is often useful.
3. Choose one variable, say x, to serve as master variable, and use the information given in the problem to express all other auxiliary variables in terms of x. Again a diagram may be helpful.
4. Use the results of steps (2) and (3) to express the given quantity to be maximized or minimized in terms of x alone.
5. Use the theory of this chapter to find the desired maximum or minimum value. This usually involves finding a derivative and determining where it is 0, and then either evaluating the given quantity at endpoints and critical points or using Theorem 4.11 or 4.12.

EXERCISES 4.5

1. Find the two positive numbers whose sum is 18 and whose product is as large as possible.
2. Find the two real numbers whose difference is 16 and whose product is as small as possible.
3. A crate open at the top has vertical sides, a square bottom, and a volume of 4 cubic meters. If the crate has the least possible surface area, find its dimensions.
4. Suppose the crate in Exercise 3 has a top. Find the dimensions of the crate with minimum surface area.
5. Show that the entire region enclosed by the outdoor track in Example 3 has maximum area if the track is circular.
6. A Norman window is a window in the shape of a rectangle with a semicircle attached at the top (Figure 4.41). Find the dimensions that allow the maximum amount of light to enter, under the condition that the perimeter of the window is 12 feet.
7. Suppose a window has the shape of a rectangle with an equilateral triangle attached at the top. Find the dimensions that allow the maximum amount of light to enter, provided that the perimeter of the window is 12 feet.
8. A rectangle is inscribed in a semicircle of radius r, with one side lying on the diameter of the semicircle. Find the maximum possible area of the rectangle.

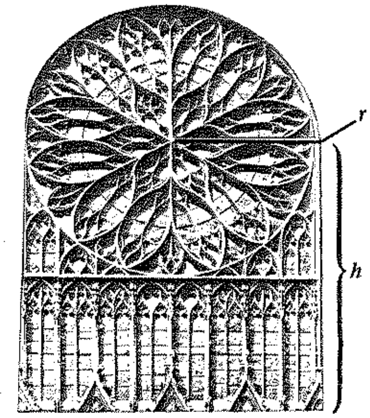


FIGURE 4.41

9. At 3 P.M. an oil tanker traveling west in the ocean at 15 kilometers per hour passes the same point as a luxury liner which arrived at the same spot at 2 P.M. while traveling north at 25 kilometers per hour. At what time were the ships closest together?
10. The coughing problem in Example 2 can be approached from a slightly different point of view. If v denotes the

velocity of the air in the windpipe, then F, r, and v are related by the equation

$$F = v(\pi r^2)$$

Consequently by (3),

$$v = \frac{F}{\pi r^2} = \frac{k}{\pi}(r_0 - r)r^2$$

Show that the velocity v is maximized when $r = \frac{2}{3}r_0$. (This shows that constriction of the windpipe during a cough appears to increase the air velocity in the windpipe and facilitate the cough.)

11. Find the point on the line $y = 2x - 4$ that is closest to the point $(1, 3)$.

12. Find the points on the parabola $y = x^2 + 2x$ that are closest to the point $(-1, 0)$.

13. Of all the triangles that pass through the point $(1, 1)$ and have two sides lying on the coordinate axes, one has the smallest area. Determine the lengths of its sides.

14. A horse breeder plans to set aside a rectangular region of 1 square kilometer for horses and wishes to build a wooden fence to enclose the region. Since one side of the region will run along a well-traveled highway, the breeder decides to make that side more attractive, using wood that costs three times as much per meter as the wood for the other sides. What dimensions will minimize the cost of the fence?

15. Suppose a landowner wishes to use 3 miles of fencing to enclose an isosceles triangular region of as large an area as possible. What should be the lengths of the sides of the triangle?

16. A manufacturer wishes to produce rectangular containers with square bottoms and tops, each container having a capacity of 250 cubic inches. If the material used for the top and the bottom costs twice as much per square inch as the material for the sides, what dimensions will minimize the cost?

17. A wire of length L is cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. Determine the minimum possible value for the sum A of the areas of the square and the circle. If the wire is actually cut, is there a maximum value of A?

18. A 12-foot wire is cut into 12 pieces, which are soldered together to form a rectangular frame whose base is twice as long as it is wide (as in Figure 4.42). The frame is then covered with paper.
    a. How should the wire be cut if the volume of the frame is to be maximized?
    b. How should the wire be cut if the total surface area of the frame is to be maximized?

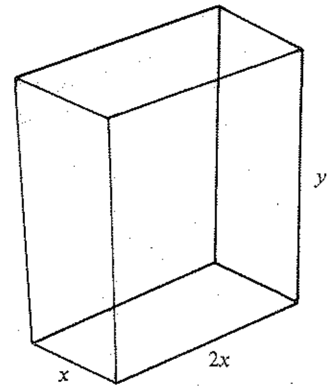


FIGURE 4.42

19. A company plans to invest \$50,000 for the next four years, and initially it buys oil stocks. If it seems profitable to do so, the oil stocks will be sold before the four-year period has lapsed, and the revenue from the sale of the stocks will be placed in tax-free municipal bonds. According to the company's analysis, if the oil stocks are sold after t years, then the net profit $P(t)$ in dollars for the four-year investment is given by

$$P(t) = 2(20 - t)^3 t \quad \text{for } 0 \le t \le 4$$

Determine whether the company should switch from oil stocks to municipal bonds, and if so, after what period of time.

20. Most post offices in the United States have the following limit on the size of a parcel that can be mailed by parcel post: The sum of the length of its longest side and its girth (the largest perimeter of a cross-section perpendicular to the longest side) can be no more than 84 inches.
    a. Find the dimensions of the rectangular parallelepiped with a square base having the largest volume that can be mailed. (There are two cases to be considered, depending on which side is longest.)
    b. Find the dimensions of the right circular cylinder having the largest volume that can be mailed. (Again, there are two cases to consider.)
    c. Find the dimensions of the cube having the largest volume that can be mailed.
    d. Show that it is possible for a parcel to be mailable and yet have a larger volume than a parcel that is not mailable. (*Hint:* Examine your solutions to (a)–(c).)
    e. For many small, rural, and army post offices, as well as post offices in Hawaii, Alaska, and Puerto Rico, the 84-inch limit is replaced by a 100-inch limit. Show that with the 100-inch limit it is still possible for a parcel to be mailable and yet have a larger volume than one that is not mailable.

21. If $C(x)$ is the cost of manufacturing an amount x of a given product and p is the price per unit amount, then the profit $P(x)$ obtained by selling an amount x is

$$P(x) = px - C(x)$$

(Notice that there is a loss if $P(x)$ is negative.)

a. If $C(x) = cx$ and $c < p$, is there a maximum profit?
b. If $C(x) = (x - 1)^2 + 2$, find the maximum profit.

22. Toward what point on the road should the ranger in Example 5 walk in order to minimize the travel time to the car if the car is located

a. 10 miles down the road?
b. $\frac{1}{2}$ mile down the road?
c. an arbitrary number c of miles down the road?

23. It is known that homing pigeons fly faster over land than over water. Assume that they fly 10 meters per second over land but only 9 meters per second over water.

a. If a pigeon is located at the edge of a straight river 500 meters wide and must fly to its nest, located 1300 meters away on the opposite side of the river (Figure 4.43), what path would minimize its flying time?
b. If the nest were located 200 meters farther down the river, what path would minimize the flying time?

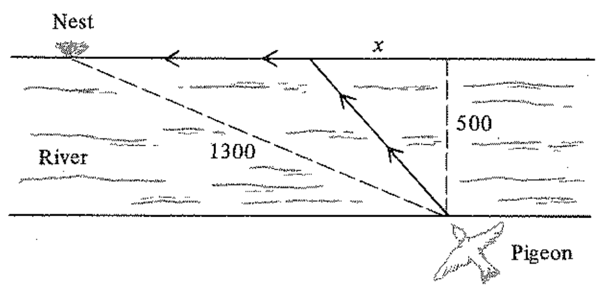


FIGURE 4.43

24. In an autocatalytic chemical reaction a substance A is converted into a substance B in such a manner that

$$\frac{dx}{dt} = kx(a - x)$$

where x is the concentration of substance B at time t, a is the initial concentration of substance A, and k is a positive constant. Determine the value of x at which the rate dx/dt of the reaction is maximum.

25. If we neglect air resistance, then the range of a ball (or any projectile) shot at an angle θ with respect to the x axis and with an initial velocity v_0 is given by

$$R(\theta) = \left(\frac{v_0^2}{g}\right)\sin 2\theta \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$$

where g is the acceleration due to gravity (32 feet per second per second).

a. Show that the maximum range is attained when $\theta = \pi/4$.
b. If $v_0 = 96$ feet per second and the aim is to snuff out a smouldering cigarette lying on the ground 144 feet away, at what angle should the ball be hit?
c. The maximum height reached by the ball is

$$y_{\max} = \frac{(v_0^2 \sin^2\theta)}{2g}$$

Why would it be a bad idea to hit the ball so that $y_{\max}$ is maximized?

26. A ring of radius a carries a uniform electric charge Q. The electric field intensity at any point x along the axis of the ring is given by

$$E(x) = \frac{Qx}{(x^2 + a^2)^{3/2}}$$

Find the maximum value of E.

27. If electric charge is uniformly distributed throughout a circular cylinder (such as a telephone wire) of radius a, then at any point whose distance from the axis of the cylinder is r, the electric field intensity is given by

$$E(r) = \begin{cases} cr & \text{for } 0 \le r \le a \\ ca^2/r & \text{for } r > a \end{cases}$$

where c is a positive constant.

a. Show that $E(r)$ is maximum for $r = a$.
b. Is E differentiable at a? Explain your answer.

28. An isosceles triangle has base 6 and height 12. Find the maximum possible area of a rectangle that can be placed inside the triangle with one side on the base of the triangle.

29. An isosceles triangle is inscribed in a circle of radius r. Find the maximum possible area of the triangle.

30. A cylindrical can with top and bottom has volume V. Find the radius of the can with the smallest possible surface area.

31. A cylinder is inscribed in a sphere with radius R. Find the height of the cylinder with the maximum possible volume.

32. A cylinder is inscribed in a cone of height H and base radius R. Determine the largest possible volume for the cylinder.

33. Find the radius of the cone with given volume V and minimum surface area. (*Hint:* The surface area S of a cone with radius r and height h is given by $S = \pi r\sqrt{r^2 + h^2}$.)

34. Three sides of a trapezoid are of equal length L, and no two are parallel. Find the length of the fourth side that gives the trapezoid maximum area.

35. A rectangular printed page is to have margins 2 inches wide at the top and the bottom and margins 1 inch wide on each of the two sides. If the page is to have 35 square inches of printing, determine the minimum possible area of the page itself.

36. A real estate firm can borrow money at 5% interest per year and can lend the money to its customers. If the amount of money it can lend is inversely proportional to the square of the interest rate at which it lends, what interest rate would maximize the firm's profit per year? (*Hint:* Let x be the loan interest rate. Notice that the profit is the product of the amount borrowed by the firm and the difference between the interest rates at which it lends and borrows.)

37. A company has a daily fixed cost of \$5000. If the company produces x units daily, then the daily cost in dollars for labor and materials is $3x$. The daily cost of equipment maintenance is $x^2/2{,}500{,}000$. What daily production minimizes the total daily cost per unit of production? (*Hint:* The cost per unit is the total cost $C(x)$ divided by x.)

38. A company sells 1000 units of a certain product annually, with no seasonal fluctuations in demand. It always reorders the same number x of units, stocks unsold units until no more remain, and then reorders again. If it costs b dollars to stock one unit for one year and there is a fixed cost of c dollars each time the company reorders, how many units should be reordered each time to minimize the total annual cost of reordering and stocking? (*Hint:* The company will have an average inventory of $x/2$ units and must reorder $1000/x$ times per year. Find the annual stocking and reordering costs and minimize their sum.)

39. Suppose we wish to estimate the probability p of rolling a 3 with a loaded die. We roll the die n times and obtain m 3's in a particular order. The probability of this is known to be $p^m(1-p)^{n-m}$. The ***maximum likelihood estimate*** of p based on the n rolls is the value of x that maximizes $x^m(1-x)^{n-m}$ on $[0, 1]$. Show that the maximum likelihood estimate of p is m/n.

°40. A farmer wishes to employ tomato pickers to harvest 62,500 tomatoes. Each picker can harvest 625 tomatoes per hour and is paid \$6 per hour. In addition, the farmer must pay a supervisor \$10 per hour and pay the union \$10 for each picker employed.
   a. How many pickers should the farmer employ to minimize the cost of harvesting the tomatoes?
   b. What is the minimum cost to the farmer?

41. Find the length of the largest thin, rigid pipe that can be carried from one 10-foot-wide corridor to a similar corridor at right angles to the first. Assume that the pipe has negligible diameter. (*Hint:* Find the length of the shortest line that touches the inside corner of the hallways and extends to the two walls.)

42. After work a person wishes to sit in a long park bounded by two parallel highways 300 meters apart. Suppose one highway is 8 times as noisy as the other. In order to have the quietest repose, how far from the quieter highway should the person sit? (*Hint:* The intensity of noise where the person sits is directly proportional to the intensity of noise at the source and inversely proportional to the square of the distance from the source.)

In Exercise 43 we present a mathematical problem that arises in two completely different settings (see Exercises 44 and 45).

43. Let p, q, and r be positive constants with $q < r$, and let

$$f(\theta) = p - q\cot\theta + \frac{r}{\sin\theta} \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

Show that f has a minimum value on $(0, \pi/2)$ at the value of θ for which $\cos\theta = q/r$.

44. This problem derives from the biological study of vascular branching. Assume that a major blood vessel A leads away from the heart (P in Figure 4.44) and that in order for the heart to feed an organ at R, we must place an auxiliary artery somewhere between P and Q. The resistance $\mathcal{R}$ of the blood as it flows along the path PSR is given by

$$\mathcal{R}(\theta) = k\left[\frac{(a - b\cot\theta)}{r_1^4}\right] + k\left(\frac{b}{r_2^4\sin\theta}\right) \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

where k, a, b, r_1, and r_2 are positive constants with $r_1 > r_2$ (see Figure 4.44). Where should the contact at S be made to produce the least resistance? (*Hint:* Using the result of Exercise 43, find the cosine of the angle θ for which $R(\theta)$ is minimized.)

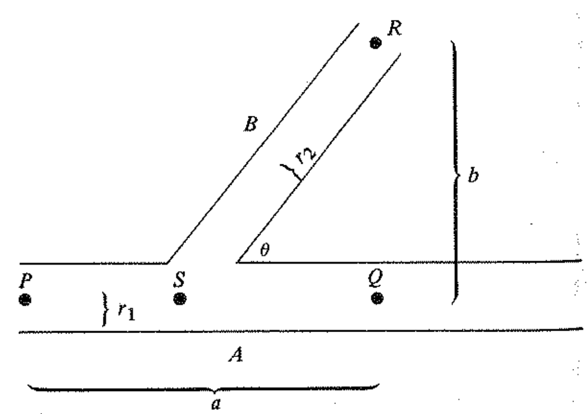


FIGURE 4.44

45. A bee's cell in a hive is a regular hexagonal prism open at the front, with a trihedral apex at the back (Figure 4.45). It can be shown that the surface area of a cell with apex θ is given by

$$S(\theta) = 6ab + \frac{3}{2}b^2\left(-\cot\theta + \frac{\sqrt{3}}{\sin\theta}\right) \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

where a and b are positive constants. Show that the surface area is minimized if $\cos\theta = 1/\sqrt{3}$, so that $\theta \approx 54.7°$. (*Hint:* Use the result of Exercise 43.) Experiments have shown that bee cells have an average angle within 2′ (less than one tenth of one degree) of 54.7°.

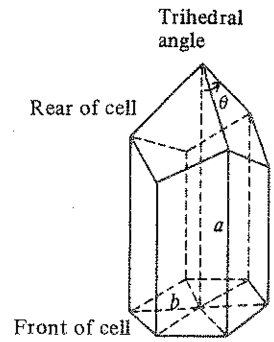
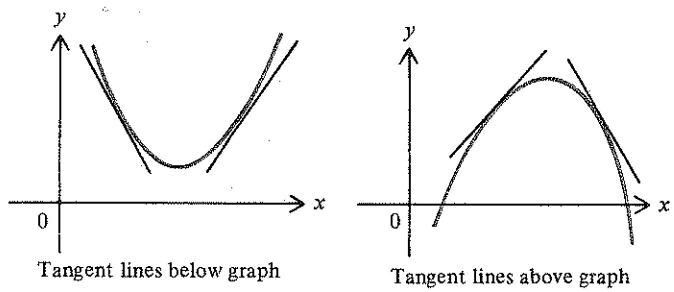


FIGURE 4.45

4.6 CONCAVITY AND INFLECTION POINTS

We now consider other ways the second derivative can help in graphing functions—through the notions of concavity and inflection points.



(a) (b)

FIGURE 4.46

Concavity

In Figure 4.46(a) the tangent lines lie below the graphs, whereas in Figure 4.46(b) the tangent lines lie above the graphs. To distinguish between these two cases, we define the notion of concavity.

DEFINITION 4.13

a. Let f be differentiable at c, and let l_c be the line tangent to the graph of f at $(c, f(c))$. The graph of f is ***concave upward*** at $(c, f(c))$ if there is an open interval I_c about c such that if x is in I_c and $x \neq c$, then $(x, f(x))$ lies above l_c. The graph of f is ***concave downward*** at $(c, f(c))$ if there is an open interval I_c about c such that if x is in I_c and $x \neq c$, then $(x, f(x))$ lies below l_c.