Math 285 Sample Exam 1 Version A

1. Solve the initial value problem $y' + 3y = e^{-3x}$ with $y(0) = 7$.

Linear ODE:

Integrating factor

$$\mu = e^{\int 3dx} = e^{3x}$$

Thus

$$y'e^{3x} + 3e^{3x}y = e^{3x} \cdot e^{-3x}$$

$$(ye^{3x})' = 1$$

$$ye^{3x} = \int 1dx = x + C$$

General solution

$$y = xe^{-3x} + Ce^{-3x}$$

Solve for constant C

$$y(0) = 0 \cdot e^{-3\cdot 0} + C \cdot e^{-3\cdot 0} = C = 7$$

Solution

$$y = (x+7)e^{-3x}$$

2. Solve $(e^{2y} - y)(\cos x)\dfrac{dy}{dx} = e^{y}\sin 2x$ with $y(0) = 0$.

Separable ODE :

$$\frac{e^{2y} - y}{e^{y}}\,dy = \frac{\sin 2x}{\cos x}\,dx$$

$$\int \frac{e^{2y}-y}{e^{y}}\,dy = \int e^{y} - ye^{-y}\,dy = e^{y} - \int ye^{-y}dy$$

$$\int ye^{-y}dy = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y}$$

$u = y \quad du = dy$

$dv = e^{-y}dy \quad v = -e^{-y}$

$$\int \frac{\sin 2x}{\cos x}\,dx = \int \frac{2\sin x\cos x}{\cos x}\,dx = 2\int \sin x\,dx$$

$$= 2\cos x + C$$

Therefore

$$e^{y} - ye^{-y} - e^{-y} = 2\cos x + C$$

Solve for C:

$$e^{0} - 0\cdot e^{-0} - e^{-0} = 2\cos 0 + C$$

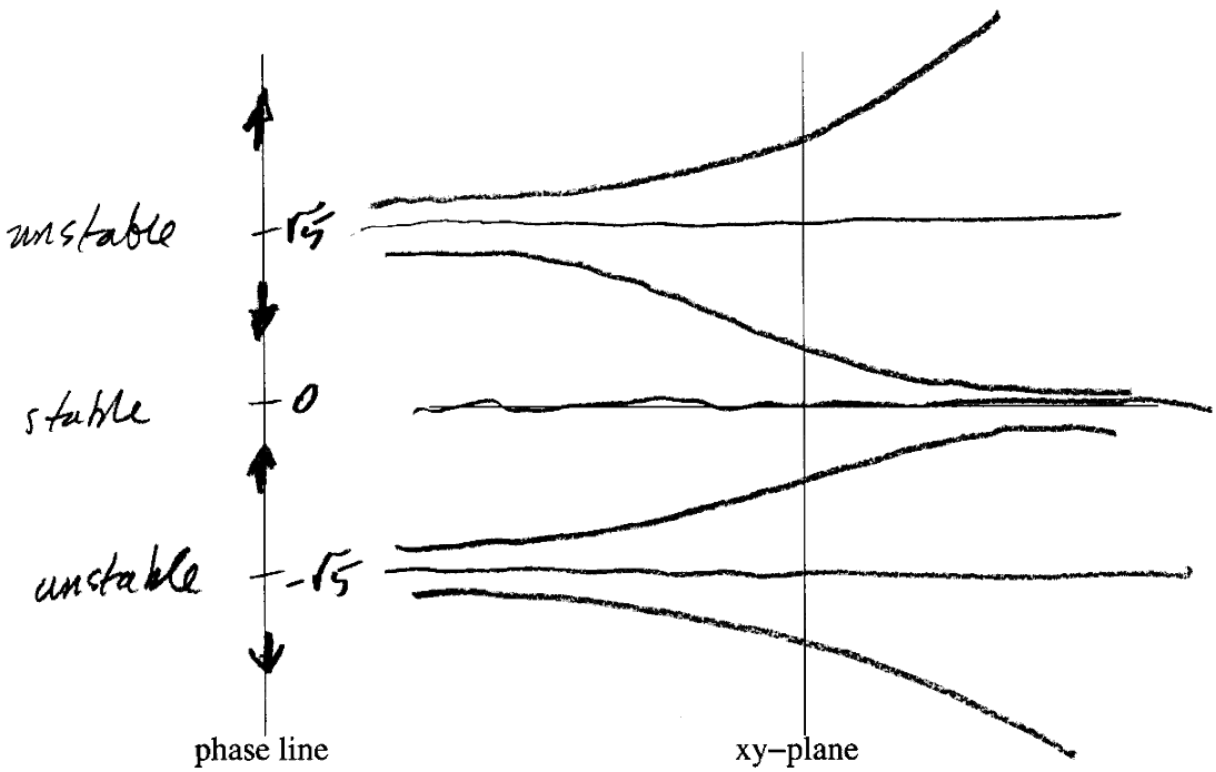
$$1 - 0 - 1 = 2 + C$$

$$C = -2$$

Solution $e^{y} - (y+1)e^{-y} = 2\cos x - 2$

3. Draw a phase portrait and solution curves for the autonoumous first-order ordinary differential equation $y' = y^3 - 5y$ below. Label the stationary points and determine whether they are stable, unstable or semi-stable.

$y^3 - 5y = 0 \qquad y(y^2-5) = 0 \qquad y = 0, \pm\sqrt{5}$



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4. Show that the ordinary differential equation

$$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y + 2)dy = 0$$

is exact and find the general solution.

$M = y\cos x + 2xe^y$ $\qquad$ $N = \cancel{\sin x} + \cancel{x^2e^y} + 2$

$M_y = \cos x + 2xe^y$ $\qquad$ $N_x = \cos x + 2xe^y$

Since $M_y = N_x$ then the ODE is exact.

Holding y constant...

$\phi = \int(y\cos x + 2xe^y)\,dx = y\sin x + x^2e^y + g(y)$

Thus

$\phi_y = \frac{\partial}{\partial y}\left(y\sin x + x^2e^y + g(y)\right) = \cancel{\sin x} + \cancel{x^2e^y} + g'(y) = N$

$$g'(y) = 2 \qquad g(y) = 2y + C$$

Thus

$\phi = y\sin x + x^2e^y + 2y + C$

The solution is

$$y\sin x + x^2e^y + 2y = C.$$

5. Find the unique solution to $\frac{dy}{dx} = \frac{2x}{1+2y}$ with $y(0) = 1$.

Separable equation

$$(1+2y)dy = 2x\,dx$$

$$\int (1+2y)dy = y + y^2$$

$$\int 2x\,dx = x^2$$

General solution

$$y + y^2 = x^2 + C$$

Solve for C

$$1 + 1^2 = 0^2 + C, \quad C = 2$$

Solution $\boxed{y + y^2 = x^2 + 2}$

simplify

$$y^2 + y + \frac{1}{4} = x^2 + \frac{9}{4}$$

$$\left(y + \frac{1}{2}\right)^2 = x^2 + \frac{9}{4}$$

$$y + \frac{1}{2} = \pm\sqrt{x^2 + 9/4}$$

Choose + solution since $y(0) = 1$.

Solution $y = -\frac{1}{2} + \sqrt{x^2 + 9/4}$.

6. Find the general solution to the differential equation

$$(y^2 + 2xy)dx - x^2 dy = 0.$$

This is homogeneous, but also a Bernoulli equation.

Solve as a homogeneous equation:

substitute $y = ux$ then $dy = du \cdot x + u\,dx$ and

$$\left((ux)^2 + 2x(ux)\right)dx - x^2(du \cdot x + u\,dx) = 0$$

$$(u^2 + 2u - u)dx = x\,du$$

Now separable

$$\int \frac{du}{u^2+u} = \int \frac{dx}{x}$$

$$\int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right)du = \ln|u| - \ln|u+1|$$

$$\int \frac{dx}{x} = \ln|x|$$

Thus $\quad \ln|u| - \ln|u+1| = \ln|x| + C$

simplify and write in terms of y and x

$$\ln\left|\frac{u}{u+1}\right| = \ln|x| + C$$

$$\left|\frac{u}{u+1}\right| = |x|e^C, \qquad \frac{u}{u+1} = Cx$$

$$u = Cxu + Cx, \qquad u(1 - Cx) = Cx$$

$$u = \frac{Cx}{1-Cx}, \qquad \frac{y}{x} = \frac{Cx}{1-Cx}$$

solution $\quad y = \dfrac{Cx^2}{1-Cx}$

Solve as a Bernoulli equation:

$$x^2 dy = (y^2 + 2xy)x$$

$$y' = \frac{y^2 + 2xy}{x^2} = \frac{2}{x}y + \frac{1}{x^2}y^2$$

so $n = 2$ and

$$u = y^{1-n} = 1/y$$

$$u' = \frac{-1}{y^2}y' = \frac{-1}{y^2}\left(\frac{2}{x}y + \frac{1}{x^2}y^2\right)$$

$$u' = -\frac{2}{x}\frac{1}{y} - \frac{1}{x^2} = -\frac{2}{x}u - \frac{1}{x^2}$$

Linear equation

$$u' + \frac{2}{x}u = -\frac{1}{x^2}$$

integrating factor

$$\mu = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$u'x^2 + 2xu = -1$$

$$(ux^2)' = -1$$

$$ux^2 = -\int 1\,dx = -x + C$$

$$u = -1/x - C/x^2$$

simplify and write in terms of x and y

$$\frac{1}{y} = -\frac{1}{x} - \frac{C}{x^2} = \frac{-x - C}{x^2}$$

$$y = \frac{x^2}{-x - C}, \quad y = \frac{x^2}{c - x} = \frac{x^2/c}{1 - x/c}$$

Thus $\quad y = \dfrac{cx^2}{1 - cx}$

7. Find the general solution to the differential equation

$$x\frac{dy}{dx} + y = x^2y^2.$$

Bernoulli equation: $n = 2$ so $u = y^{1-n} = 1/y$.

$$u' = -\frac{1}{y^2}y' = -\frac{1}{y^2}\left(-\frac{y}{x} + xy^2\right) = \frac{1}{x}\cdot\frac{1}{y} - x$$

$$u' = \frac{1}{x}u - x, \quad u' - \frac{1}{x}u = -x$$

linear equation, integrating factor

$$\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$u'\frac{1}{x} - \frac{1}{x^2}u = -1$$

$$\left(u\frac{1}{x}\right)' = -1$$

$$u\frac{1}{x} = -\int 1\,dx = -x + C$$

$$u = -x^2 + Cx$$

Simplify and write in terms of x and y

$$\frac{1}{y} = -x^2 + Cx$$

$$y = \frac{1}{-x^2 + Cx}$$

$$y = \frac{1}{Cx - x^2}$$