Math 330 Homework 7 Version A

**1.** Consider the quadratic form $X^t AX$ where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

**(i)** Solve the eigenvalue-eigenvector problem $AV = \lambda V$.

**(ii)** Find an orthogonal matrix $Q$ such that $Q^t AQ$ is diagonal.

**2.** Use the Gram–Schmidt algorithm to find a set of orthonormal vectors that span the same space as the given vectors.

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Do the calculations by hand.

**3.** Use the Maple subroutine `GramSchmidt` to find a set of orthonormal vectors that span the same space as the vectors.

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 7 \\ -1 \\ 3 \\ 8 \\ 6 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

**4.** Construct a matrix $A$ which has the eigenvectors

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad V_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

with the corresponding eigenvalues

$$\lambda_1 = 1, \qquad \lambda_2 = -2, \qquad \lambda_3 = 5.$$

**5.** Find the eigenvectors and the eigenvalues by hand for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

**6.** Prove that any set of orthonormal vectors is a linearly independent set.

**7.** The following output shows how to use the Maple subroutine `Eigenvectors` to find the eigenvectors and eigenvalues of a matrix.

```
> restart;
> with(LinearAlgebra);
> A:=Matrix([[1,2],[0,4]]);
```

$$A := \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2/3 \\ 0 & 1 \end{bmatrix}$$

Interpret this output and explicitly write down the eigenvectors $V_1$ and $V_2$ and the corresponding eigenvalues $\lambda_1$ and $\lambda_2$.

**8.** Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

**(i)** How many eigenvalues does $A$ have?

**(ii)** Find all the eigenvectors of $A$.

**(iii)** Extra Credit: Prove $A$ has only one linearly independent eigenvector.

**(iv)** Find the singular values of $A$.

**(v)** Extra Credit: Find the singular value decomposition $A = U\Sigma V^t$ where $U$ and $V$ are orthogonal matrices and $\Sigma$ is diagonal.

**9.** Consider a non-singular matrix $A \in M_{n\times n}$ of the form

$$A = \Big[ X_1 \Big| \cdots \Big| X_n \Big] \qquad \text{where} \qquad X_i \in \mathbf{R}^n.$$

Apply the Gram–Schmidt procedure to the columns of $A$ to obtain the orthonormal set of vectors $\{V_1, \ldots, V_n\}$ and define $R = Q^t A$ where $Q$ is the orthogonal matrix

$$Q = \Big[ V_1 \Big| \cdots \Big| V_n \Big].$$

**(i)** Show that $A = QR$.

**(ii)** Find $Q$ and $R$ for the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

**(iii)** Extra Credit: Prove that $R$ is always upper triangular.