§3.4 Find the complete solution to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

First do elimination on the augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \quad r_2 \leftarrow r_2 - 2r_1 \quad \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix}$$

$$r_3 \leftarrow r_3 - r_1 \quad \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equations are

$$x_1 + 3x_2 + x_3 + 2x_4 = 1$$
$$2x_3 + 4x_4 = 1$$

Solve and back substitution

$$x_3 = \frac{1-4x_4}{2} = \frac{1}{2} - 2x_4$$

$$x_1 = 1 - 3x_2 - x_3 - 2x_4 = 1 - 3x_2 - \frac{1-4x_4}{2} - 2x_4$$
$$= \frac{1}{2} - 3x_2$$

The solution is

$$x = \underbrace{\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{x_n} \quad \text{where } x_2, x_4 \in \mathbb{R}.$$