Math 330 Final Exam Version A

1. Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 3 & 1 & 6 \end{bmatrix}, \qquad x = \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix}.$$

(i) Find $\frac{1}{3}x$

(ii) Find $x + b$

(iii) Find $x \cdot b$

(iv) Find $\|b\|$

(v) Find Ax

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Write A as LDU where L is lower triangular with ones on its diagonal, D is diagonal and U is upper triangular with ones on its diagonal.

3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Find the reduced row echelon form R of A.

4. Consider the matrix A with reduced row echelon form R given by

$$A = \begin{bmatrix} 6 & 0 & 6 & 1 & 4 \\ 0 & 3 & 6 & 3 & 0 \\ 1 & 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 4 & 5 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(i) Find a basis for the subspace $\mathcal{C}(A)$ and state its dimension.

(ii) Find a basis for the subspace $\mathcal{N}(A)$ and state its dimension.

5. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Find $\det A$.

6. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

7. Let

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2\sqrt{2}}{3} & \frac{-1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Note that Q is orthogonal and R upper triangular. Suppose $A = QR$. Find the x which minimizes $\|Ax - b\|$.

8. Let
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$
Find the eigenvectors and eigenvalues of A

9. Let
$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}.$$
Find the singular value decomposition $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal.

10. Let $u, v \in \mathbf{R}^2$ and θ be the angle between u and v. Show that $u \cdot v = \|u\|\|v\| \cos\theta$.

11. Let $A \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{l \times m}$.

(i) Show that $\mathcal{C}(BA) \subseteq \mathcal{C}(B)$.

(ii) Given a concrete example where $\mathcal{C}(B) \neq \mathcal{C}(BA)$.

12. Let $A \in \mathbf{R}^{m \times n}$ where $m \neq n$. Suppose that the rank of A is $\operatorname{rank}(A) = r$.

(i) What is $\dim \mathcal{C}(A)$?

(ii) What is $\dim \mathcal{N}(A)$?

(iii) What is $\dim \mathcal{C}(A)^{\perp}$?

(iv) What is $\dim \mathcal{N}(A)^{\perp}$?

(v) Show that $\mathcal{C}(A)^{\perp} = \mathcal{N}(A^T)$.

13. Let $A \in \mathbf{R}^{4\times 7}$.

(i) Find the matrix E such that AE corresponds to the result obtained after performing the column operation $c_2 \leftarrow c_2 - 3c_1$ on the matrix A.

(ii) Prove or disprove the claim that $\mathcal{N}(AE) = \mathcal{N}(A)$. If true explain why; if false provide a counterexample.