~ Key ~

Math 330 Final Exam Version B

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}, \qquad x = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(i) Find $\frac{2}{3}x$ $= \frac{2}{3}\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2/3 \\ -2/3 \end{bmatrix}$

(ii) Find $x + b$ $= \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

(iii) Find $x \cdot b$ $= \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 3 - 2 - 1 = 0$

(iv) Find $\|b\|$ $= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

(v) Find Ax $= \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3-2-3 \\ -3+1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$

2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Find the reduced row echelon form R of A.

$r_2 \leftarrow r_2 - 3r_1$
$r_3 \leftarrow r_3 - r_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$r_2 \leftrightarrow r_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

3. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -3 & 6 \\ 6 & 5 & -9 \end{bmatrix}.$$

Write A as LDU where L is lower triangular with ones on its diagonal, D is diagonal and U is upper triangular with ones on its diagonal.

$r_2 \leftarrow r_2 + 2r_1$
$r_3 \leftarrow r_3 - 3r_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 2 & -9 \end{bmatrix}$$

$r_3 \leftarrow r_3 + 2r_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$r_3 \leftarrow \frac{1}{3} r_3$
$r_2 \leftarrow -r_2$
$r_1 \leftarrow \frac{1}{2} r_1$

$$U = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

4. Consider the matrix A with reduced row echelon form R given by

$$A = \begin{array}{c} \begin{array}{cccccc} F & P & P & F & F & P \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 8 & -1 & 3 \\ 0 & -1 & 0 & -2 & 5 & 2 \\ 0 & 3 & -1 & 3 & -17 & -1 \\ 0 & 1 & 1 & 5 & -3 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad R = \begin{bmatrix} 0 & 1 & 0 & 2 & -5 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find a basis for the subspace $\mathcal{C}(A)$ and state its dimension.

Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}$

$\dim \mathcal{C}(A) = 3$

(ii) Find a basis for the subspace $\mathcal{N}(A)$ and state its dimension.

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim \mathcal{N}(A) = 3$

5. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} = [r_3 \leftrightarrow r_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Find $\det A$.

$$\det A = \det [r_3 \leftrightarrow r_4] \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = (-1)\, 1 \cdot 2 \cdot 3 \cdot 4$$

$= -24.$

6. Let $A \in \mathbf{R}^{m \times n}$ where $m \neq n$. Suppose that the rank of A is $\operatorname{rank}(A) = r$.

(i) What is $\dim \mathcal{C}(A)$?

r

(ii) What is $\dim \mathcal{N}(A)$?

$n - r$

(iii) What is $\dim \mathcal{C}(A)^{\perp}$?

$m - r$

(iv) What is $\dim \mathcal{N}(A)^{\perp}$?

r

(v) Show that $\mathcal{C}(A)^{\perp} = \mathcal{N}(A^T)$.

$$
\begin{aligned}
\mathcal{C}(A)^{\perp} &= \{ y \in \mathbb{R}^m : v \cdot y = 0 \text{ for every } v \in \mathcal{C}(A) \} \\
&= \{ y \in \mathbb{R}^m : Ax \cdot y = 0 \text{ for every } x \in \mathbb{R}^n \} \\
&= \{ y \in \mathbb{R}^m : (Ax)^T y = 0 \text{ for every } x \in \mathbb{R}^n \} \\
&= \{ y \in \mathbb{R}^m : x^T A^T y = 0 \text{ for every } x \in \mathbb{R}^n \} \\
&= \{ y \in \mathbb{R}^m : x \cdot A^T y = 0 \text{ for every } x \in \mathbb{R}^n \} \\
&= \{ y \in \mathbb{R}^m : A^T y = 0 \} = \mathcal{N}(A^T).
\end{aligned}
$$

7. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

$\tilde{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\tilde{q}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - q_1\left(q_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\quad Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$R = Q^T A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$

Check:

$QR = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$

8. Let

$$Q = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Note Q is a matrix with orthonormal columns and R is upper triangular. Suppose $A = QR$. Find the x which minimizes $\|Ax - b\|$.

Solve: $Q^T A x = Q^T b$ or $Rx = Q^T b$.

$Q^T b = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

back substitution:

$x_2 = 0$, $x_1 = 2 - 0 = 2$

Therefore $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ minimizes $\|Ax - b\|$.

9. Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{l\times m}$. Show that $(BA)^T = A^T B^T$.

Since $\underset{l\times m\ m\times n}{BA}$ is $l\times n$ matrix, then $(BA)^T$ is $n\times l$.

Let $i=1,\dots,n$ and $j=1,\dots,l$. Then

$$(BA)^T_{ij} = (BA)_{ji} = \sum_{k=1}^{m} B_{jk}A_{ki} = \sum_{k=1}^{m}(B^T)_{kj}(A^T)_{ik}$$

$$= \sum_{k=1}^{m}(A^T)_{ik}(B^T)_{kj} = A^TB^T.$$

10. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find the eigenvectors and eigenvalues of A

$\det(A-\lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$ so $\lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$.

$\lambda_1=1$

$$A-\lambda_1 I = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad r_2 \leftarrow r_2 - \tfrac{1}{2}r_1 \quad \begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{matrix} r_3 \leftarrow r_3 + 2r_1 \\ r_1 \leftarrow r_1 + 4r_1 \end{matrix} \quad \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} r_1 \leftarrow \tfrac{1}{2}r_1 \\ r_2 \leftarrow -r_2 \end{matrix} \quad R_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2=2$

$$A-\lambda_2 I = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} r_3 \leftarrow r_3 - r_2 \\ r_1 \leftarrow r_1 - 4r_2 \end{matrix} \quad \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_1 \leftarrow r_1 \quad R_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3=3$

$$A-\lambda_3 I = \begin{bmatrix} -2 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad r_1 \leftarrow r_1 + 2r_2 \quad \begin{bmatrix} -2 & 0 & 6 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} r_1 \leftarrow -\tfrac{1}{2}r_1 \\ r_2 \leftarrow -r_2 \end{matrix} \quad R_3 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

11. Let $A \in \mathbf{R}^{4\times 4}$ be symmetric so $A^T = A$. Show that the eigenvalues of A are real.

Let λ be an eigenvalue with eigenvector x. Thus $Ax = \lambda x$. Then

$$\lambda x \cdot \bar{x} = Ax \cdot \bar{x} = (Ax)^T \bar{x} = x^T A^T \bar{x} = x \cdot A^T \bar{x} = x \cdot A\bar{x} = x \cdot \bar{A}\bar{x}$$

$$= x \cdot \overline{Ax} = x \cdot \overline{\lambda x} = x \cdot \bar{\lambda}\bar{x} = \bar{\lambda}\, x \cdot \bar{x}.$$

Since x is an eigenvector $x \cdot \bar{x} = \|x\|^2 \neq 0$.

Therefore $\lambda = \bar{\lambda}$. Therefore λ is real.

12. Let

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}.$$

Find the singular value decomposition $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal. $A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma U^T U \Sigma V^T = V\Sigma^2 V^T$

$$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = (5-\lambda)(8-\lambda) - 4 = 40 - 13\lambda + \lambda^2 - 4 = \lambda^2 - 13\lambda + 36$$

$$= (\lambda - 9)(\lambda - 4) \quad \text{so } \lambda_1 = 9 \text{ and } \lambda_2 = 4$$

$\lambda_1 = 9$

$$A^TA - \lambda_1 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \quad r_2 \leftarrow r_2 + \tfrac{1}{2} r_1 \quad \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \quad r_1 \leftarrow -\tfrac{1}{4} r_1 \quad \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}, \; x_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

$\lambda_2 = 4$

$$A^TA - \lambda_2 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad r_2 \leftarrow r_2 - 2r_1 \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$V = \left[\frac{x_1}{\|x_1\|} \,\Big|\, \frac{x_2}{\|x_2\|}\right] = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$U = AV\Sigma^{-1} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

check

$$U\Sigma V^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 & 10 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}.$$

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13. [Extra Credit] Find $\sin(A)$ where

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ $$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ $S = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

is the same matrix as appearing in one of the previous problems.

$AS = SD$ $\quad A = SDS^{-1}$ $\quad \sin A = S(\sin D)S^{-1}$

$$\sin D = \begin{bmatrix} \sin 1 & 0 & 0 \\ 0 & \sin 2 & 0 \\ 0 & 0 & \sin 3 \end{bmatrix}$$

Find S^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} r_2 \leftarrow r_2 - r_3 \\ r_1 \leftarrow r_1 - 3r_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$r_1 \leftarrow r_1 - 2r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad S^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sin A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin 1 & 0 & 0 \\ 0 & \sin 2 & 0 \\ 0 & 0 & \sin 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin 1 & 2\sin 2 & 3\sin 3 \\ 0 & \sin 2 & \sin 3 \\ 0 & 0 & \sin 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin 1 & -2\sin 1 + 2\sin 2 & -\sin 1 - 2\sin 2 + 3\sin 3 \\ 0 & \sin 2 & -\sin 2 + \sin 3 \\ 0 & 0 & \sin 3 \end{bmatrix}$$