Math 330 Midterm Review Version A

1. **Know how to solve all homework and quiz problems.**

2. Let $u, v \in \mathbf{R}^2$ and θ be the angle between u and v. Show that $u \cdot v = \|u\|\|v\| \cos\theta$.

3. Let $A \in \mathbf{R}^{m\times n}$.

(i) Show that the nullspace $\mathcal{N}(A)$ is a subspace of $\mathbf{R}^n$.

(ii) Show that the column space $\mathcal{C}(A)$ is a subspace of $\mathbf{R}^m$.

4. Let $A \in \mathbf{R}^{5\times 5}$ be an invertible matrix and R the reduced eschelon form of A. Is this enough information to tell what R is? If so, explain and find R, otherwise explain why this is not enough information to find R.

5. Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{l\times m}$. Prove that $(BA)^T = A^T B^T$.

6. Let $E_1, E_2 \in \mathbf{R}^{m\times m}$ be row operations of the form

$$E_1 = [r_i \leftarrow r_i + \alpha_1 r_j] \qquad \text{where} \qquad i \neq j$$

and

$$E_2 = [r_k \leftarrow r_k + \alpha_2 r_l] \qquad \text{where} \qquad k \neq l.$$

Is it always true that $E_1E_2 = E_2E_1$? If true explain why; if not provide a counter example where it is false.

7. Let $A \in \mathbf{R}^{4\times 7}$. Suppose A is a matrix that can be put into eschelon form U using elimination without pivoting. In general how may row operations $r_i \leftarrow r_i + \alpha r_j$ does the elimination algorithm take to put A into eschelon form? Write these row operations in order.

8. Suppose $A \in \mathbf{R}^{m\times n}$ and that $\mathcal{N}(A) = \{\,0\,\}$. Show $Au = Av$ implies $u = v$ for every $u, v \in \mathbf{R}^n$.

9. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(i) Find Av.

(ii) Find $v^T v$.

(iii) Find $\|v\|$.

(iv) Find $vv^T - 2A + I$.

10. Let

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Find E^{-1}.

11. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

Find A^{-1}.

12. Let A be a 3×3 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathcal{N}(A) = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} c : c \in \mathbf{R}^2 \right\}.$$

(i) Find all solutions x to $Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(ii) Find A.

13. Let

$$A = \begin{bmatrix} 1/2 & 1 & 0 & 0 & 1 & 5/2 \\ 5 & 10 & 6 & 0 & 4 & 37 \\ -1 & -2 & 1 & 2 & 5 & -2 \\ 3 & 6 & -2 & 1 & 12 & 23/2 \end{bmatrix}.$$

The reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 4 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find the rank of A.

(ii) Find a basis for the nullspace $\mathcal{N}(A)$. What is the dimension $\dim \mathcal{N}(A)$?

(iii) Find a basis for the column space $\mathcal{C}(A)$. What is the dimension $\dim \mathcal{C}(A)$?

14. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find a lower triangular matrix L and an upper triangular U such that $LU = A$.

15. Answer the following questions about linear independence, span and basis. Give a reason for each of your answers.

(i) Are the vectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

linearly independent?

(ii) Does the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ span the same space as the vector $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$?

(iii) Do the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

span the space $\mathbf{R}^3$?

16. True or false with a counterexample if false and a reason if true.

(i) If A is an $n \times n$ matrix then $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^T)$.

(ii) If A is an $m \times n$ matrix and B is an invertible $n \times n$ matrix then $\mathcal{C}(AB) = \mathcal{C}(A)$.

(iii) If P is an $n \times n$ permutation matrix then $P^n = I$.

(iv) If R is the reduced row echelon form of A then $\mathcal{C}(A^T) = \mathcal{C}(R^T)$.

17. Let $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve the system

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find x.

18. Determine whether the following pairs of vectors are perpendicular

(i) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ perpendicular to $\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$?

(ii) Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ perpendicular to $\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$?

19. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}.$$