Math 330 Quiz 3 Version A

1. The orthogonal complement $S^{\perp}$ of a subspace $S \subset \mathbf{R}^m$ is defined

(A) $S^{\perp} = \{\, v \cdot y : v \in \mathbf{R}^m \text{ and } y \in S \,\}$.

(B) $S^{\perp} = \{\, y \in \mathbf{R}^m : v \cdot y = 0 \text{ for all } v \in S \,\}$.

(C) $S^{\perp} = \{\, y \in \mathbf{R}^m : v \cdot y = 0 \text{ for at least one } y \in S \,\}$.

(D) both (B) and (C).

(E) none of these.

2. Let $A \in \mathbf{R}^{m\times n}$. Show that $\mathcal{C}(A)^{\perp} = \mathcal{N}(A^T)$.

$$\mathcal{C}(A)^{\perp} = \{ y \in \mathbb{R}^m : v \cdot y = 0 \text{ for all } v \in \mathcal{C}(A) \}$$

$$= \{ y \in \mathbb{R}^m : Ax \cdot y = 0 \text{ for all } x \in \mathbb{R}^n \}$$

$$= \{ y \in \mathbb{R}^m : (Ax)^T y = 0 \quad \text{——''——} \}$$

$$= \{ y \in \mathbb{R}^m : x^T A^T y = 0 \quad \text{——''——} \}$$

$$= \{ y \in \mathbb{R}^m : x \cdot A^T y = 0 \text{ for all } x \in \mathbb{R}^n \}$$

$$= \{ y \in \mathbb{R}^m : A^T y = 0 \} = \mathcal{N}(A^T).$$

Math 330 Quiz 3 Version A

3. Suppose $A \in \mathbf{R}^{m\times n}$ is factored as $A = QR$ where $Q \in \mathbf{R}^{m\times n}$ is has orthonormal columns and $R \in \mathbf{R}^{n\times n}$ is upper triangular. If

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \qquad R = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

then find the point $x \in \mathbf{R}^n$ that minimizes the norm $\|Ax - b\|$.

$$Ax - b \in C(A)^\perp = C(Q)^\perp = N(Q^T)$$

$$Q^T A x = Q^T b$$

$$Q^T b = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -3/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -\frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -\frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}$$

$$3x_1 + 2x_2 = \frac{1}{\sqrt{2}}$$

$$x_2 = -\frac{\sqrt{3}}{\sqrt{2}}$$

$$x_1 = \frac{\frac{1}{\sqrt{2}} - 2x_2}{3} = \frac{\frac{1}{\sqrt{2}} + \frac{2\sqrt{3}}{\sqrt{2}}}{3}$$

$$= \frac{1 + 2\sqrt{3}}{3\sqrt{2}}$$

Math 330 Quiz 3 Version A

4. Find an orthonormal basis for the space spanned by the vectors

$$\left\{ \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

$$\tilde{q}_1 = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{q}_2 = a_2 - q_1(q_1 \cdot a_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{q}_3 = a_3 - q_1(q_1 \cdot a_3) - q_2(q_2 \cdot a_3) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

$$q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{\sqrt{11}}\begin{bmatrix} 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

The orthonormal basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{11} \\ 3/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix} \right\}.$$

Math 330 Quiz 3 Version A

5. Let $A \in \mathbf{R}^{m\times n}$ where $m \neq n$. Let

$$v \in \mathcal{N}(A^T), \quad w \in \mathcal{C}(A^T), \quad x \in \mathcal{N}(A) \quad \text{and} \quad y \in \mathcal{C}(A).$$

(i) How many components does the vector v have?

m

(ii) How many components does the vector w have?

n

(iii) How many components does the vector x have?

n

(iv) Home many components does the vector$y$ have?

m

(v) What is $v \cdot y$? Since $v \in N(A^T) = C(A)^\perp$ and $y \in C(A)$ then $v \cdot y = 0$,

(vi) What is Ax? Since $x \in N(A)$ then $Ax = 0$.

(vii) Given any vector $z \in \mathbf{R}^n$ is it true that $z = v + y$ for some $v \in \mathcal{N}(A^T)$ and $y \in \mathcal{C}(A)$? Explain your answer.

No, because $v \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$, Since $m \neq n$ then these could never add to a vector in $\mathbb{R}^n$.