Orthogonal Complements

The orthogonal complement $S^\perp$ of a subspace S of $\mathbf{R}^m$ is defined

$$S^\perp = \{\, y \in \mathbf{R}^m : v \cdot y = 0 \text{ for all } v \in S \,\}.$$

If $v \in S$ then $y \cdot v = 0$ for every $y \in S^\perp$. Thus $S \subseteq (S^\perp)^\perp$.

Let A consist of columns that form a basis of S. Then $S = \mathcal{C}(A)$ and

$$\begin{aligned}
\mathcal{C}(A)^\perp &= \{\, y \in \mathbf{R}^m : v \cdot y = 0 \text{ for all } v \in \mathcal{C}(A) \,\} \\
&= \{\, y \in \mathbf{R}^m : Ax \cdot y = 0 \text{ for all } x \in R^n \,\} \\
&= \{\, y \in \mathbf{R}^m : x \cdot A^T y = 0 \text{ for all } x \in R^n \,\} \\
&= \{\, y \in \mathbf{R}^m : A^T y = 0 \,\} = \mathcal{N}(A^T).
\end{aligned}$$

Given $v \in \mathcal{N}(A^T)^\perp$, consider the $m \times (n+1)$ matrix

$$B = \left[\, A \,\middle|\, v \,\right].$$

Since $A^T y = 0$ implies $v \cdot y = 0$, then

$$\begin{aligned}
\mathcal{N}(B^T) &= \{\, y \in \mathbf{R}^m : B^T y = 0 \,\} \\
&= \{\, y \in \mathbf{R}^m : A^T y = 0 \text{ and } v \cdot y = 0 \,\} \\
&= \{\, y \in \mathbf{R}^m : A^T y = 0 \,\} = \mathcal{N}(A^T).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dim \mathcal{C}(A) = \dim \mathcal{C}(A^T) &= m - \dim \mathcal{N}(A^T) \\
&= m - \dim \mathcal{N}(B^T) = \dim \mathcal{C}(B^T) = \dim \mathcal{C}(B).
\end{aligned}$$

Since $\mathcal{C}(A) \subseteq \mathcal{C}(B)$, then $\mathcal{C}(A) = \mathcal{C}(B)$. Therefore $v \in \mathcal{C}(A)$, and so $\mathcal{N}(A^T)^\perp \subseteq \mathcal{C}(A)$. Now

$$(S^\perp)^\perp = (\mathcal{C}(A)^\perp)^\perp = \mathcal{N}(A^T)^\perp \subseteq \mathcal{C}(A) = S \subseteq (S^\perp)^\perp$$

implies $(S^\perp)^\perp = S$, and in particular $\mathcal{N}(A^T)^\perp = \mathcal{C}(A)$.

Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix. Then

$$\dim \mathcal{C}(A) = \dim \mathcal{C}(A^T) = r, \quad \dim \mathcal{N}(A) = n - r \quad \text{and} \quad \dim \mathcal{N}(A^T) = m - r.$$

Moreover,

$$\mathcal{C}(A)^\perp = \mathcal{N}(A^T), \quad \mathcal{C}(A^T)^\perp = \mathcal{N}(A), \quad \mathcal{C}(A) = \mathcal{N}(A^T)^\perp \quad \text{and} \quad \mathcal{C}(A^T) = \mathcal{N}(A)^\perp.$$