Math 330 homework 4

1. Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 3 & 1 & 6 \end{bmatrix}, \qquad x = \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix}.$$

(i) Find $\frac{1}{3}x = \frac{1}{3}\begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 \\ -1/3 \end{bmatrix}$

(ii) Find $x + b = \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$

(iii) Find $x \cdot b = \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix} = 0 + 14 - 8 = 6$

(iv) Find $\|b\| = \sqrt{0^2 + 2^2 + 8^2} = \sqrt{68}$

(v) Find $Ax = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + 0 - 1 \\ -2 + 0 + 2 \\ -3 + 7 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

Math 330 homework 4

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Write A as LDU where L is lower triangular with ones on its diagonal, D is diagonal and U is upper triangular with ones on its diagonal.

$r_2 - 2r_1$
$r_3 - r_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$ $r_3 + r_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Find the reduced row echelon form R of A.

$r_2 - 2r_1$
$r_1 - r_2$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$ $r_2 \leftrightarrow r_3$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$r_1 - r_2$ $\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Math 330 homework 4

4. Consider the matrix A with reduced row echelon form R given by

$$A = \begin{array}{c} \begin{matrix} P & P & F & P & P \end{matrix} \\ \begin{bmatrix} 6 & 0 & 6 & 1 & 4 \\ 0 & 3 & 6 & 3 & 0 \\ 1 & 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 4 & 5 \end{bmatrix} \end{array} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(i) Find a basis for the subspace $\mathcal{C}(A)$ and state its dimension.

Basis: $\left\{ \begin{bmatrix} 6 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \\ 5 \end{bmatrix} \right\}$

$\dim C(A) = 4$

(ii) Find a basis for the subspace $\mathcal{N}(A)$ and state its dimension.

Basis: $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\quad \dim N(A) = 1$

5. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Find det A.

$$\det A = \det [r_2 \leftrightarrow r_3] \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & \\ 0 & 0 & 4 & \end{bmatrix}$$

$$= (-1)\ (1 \cdot 3 \cdot 2 \cdot 4) = -24$$

Math 330 homework 4

6. Let
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

$$\tilde{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\tilde{q}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - q_1\left(q_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$R = Q^T A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

7. Let
$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2\sqrt{2}}{3} & \frac{-1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Note that Q is orthogonal and R upper triangular. Suppose $A = QR$. Find the x which minimizes $\|Ax - b\|$.

$$Q^T A x = Q^T b \quad \text{or} \quad Rx = Q^T b$$

$$Q^T b = \begin{bmatrix} 1/3 & 0 & 2\sqrt{2}/3 \\ 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} & -1/3\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

back substitution...

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$x_3 = 2$

$x_2 = 2 - 2 = 0$

$x_1 = 1 - 0 - 2 = -1$

Therefore $x = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ minimizes $\|Ax - b\|$.

8. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find the eigenvectors and eigenvalues of A

$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$ so $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

$\lambda_1 = 1$ // $A - \lambda_1 I = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ Thus $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 1$ // $A - \lambda_2 I = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{matrix} r_3 - r_2 \\ r_1 - 4r_2 \end{matrix}$ $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $r_1 \leftarrow -r_1$ $\begin{array}{ccc} P & F & P \\ \end{array}$ $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Thus $x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 3$ // $A - \lambda_3 I = \begin{bmatrix} -2 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} r_1 \leftarrow -\frac{1}{2} r_1 \\ r_2 \leftarrow -1 r_2 \end{matrix}$ $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $r_1 + r_2$ $\begin{array}{ccc} P & P & F \\ \end{array}$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Thus $x_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}.$$

Find the singular value decomposition $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal.

$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma U^T U \Sigma V^T = V\Sigma^2 V^T$

$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$

$\det(A^TA - \lambda I) = (5-\lambda)(8-\lambda) - 4 = 40 - 13\lambda + \lambda^2 - 4 = \lambda^2 - 13\lambda + 36$

$= (\lambda - 9)(\lambda - 4)$ so $\lambda_1 = 9$ and $\lambda_2 = 4$

$\lambda_1 = 9$ // $A^TA - \lambda_1 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ $r_2 \leftarrow r_2 + \frac{1}{2} r_1$ $\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$ $r_1 \leftarrow \frac{1}{4} r_1$ $\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$, $x_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$.

$\lambda_2 = 4$ // $A^TA - \lambda_2 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $r_2 \leftarrow r_2 - 2r_1$ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$V = \left[\frac{x_1}{\|x_1\|} \middle| \frac{x_2}{\|x_2\|} \right] = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

$U = AV\Sigma^{-1} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$

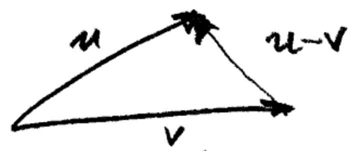
$= \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

check //

$U\Sigma V^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 & 10 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$.

Math 330 homework 4

10. Let $u, v \in \mathbf{R}^2$ and θ be the angle between u and v. Show that $u \cdot v = \|u\|\|v\|\cos\theta$.



Law of cosines:

$$\|u-v\|^2 = \cancel{\|u\|^2} + \cancel{\|v\|^2} - 2\|u\|\|v\|\cos\theta$$

Definition of norm: $\|u-v\|^2 = (u-v)\cdot(u-v) = u\cdot u - v\cdot u - u\cdot v + v\cdot v$

$$= \cancel{\|u\|^2} - 2u\cdot v + \cancel{\|v\|^2}$$

Therefore $u\cdot v = \|u\|\,\|v\|\cos\theta$.

11. Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{l\times m}$.

(i) Show that $\mathcal{C}(BA) \subseteq \mathcal{C}(B)$.

Let $y \in C(BA)$. Then there is $x \in \mathbb{R}^n$ such that $y = BAx$. Let $z = Ax$. Then $z \in \mathbb{R}^m$ and $y = Bz$ implies $y \in C(B)$.

Therefore $C(BA) \subseteq C(B)$.

(ii) Given a concrete example where $\mathcal{C}(B) \neq \mathcal{C}(BA)$. Let $m = n = l = 0$.

Suppose $A = [0]$ and $B = [1]$.

Then $C(B) = \mathbb{R}$ and $\dim C(B) = 1$

However $C(AB) = C(0) = \{0\}$

and $\dim C(AB) = 0$.

Thus $C(B) \neq C(AB)$.

Math 330 homework 4

1. Let $A \in \mathbf{R}^{m \times n}$ where $m \neq n$. Suppose that the rank of A is $\operatorname{rank}(A) = r$.

(i) What is $\dim \mathcal{C}(A)$?

r

(ii) What is $\dim \mathcal{N}(A)$?

$n - r$

(iii) What is $\dim \mathcal{C}(A)^{\perp}$?

$m - r$

(iv) What is $\dim \mathcal{N}(A)^{\perp}$?

r

(v) Show that $\mathcal{C}(A)^{\perp} = \mathcal{N}(A^T)$.

$$\begin{aligned}
\mathcal{C}(A)^{\perp} &= \{y \in \mathbb{R}^m : v \cdot y = 0 \text{ for every } v \in \mathcal{C}(A)\} \\
&= \{y \in \mathbb{R}^m : Ax \cdot y = 0 \text{ for every } x \in \mathbb{R}^n\} \\
&= \{y \in \mathbb{R}^m : (Ax)^T y = 0 \text{ for every } x \in \mathbb{R}^n\} \\
&= \{y \in \mathbb{R}^m : x^T A^T y = 0 \text{ for every } x \in \mathbb{R}^n\} \\
&= \{y \in \mathbb{R}^m : x \cdot A^T y = 0 \text{ for every } x \in \mathbb{R}^n\} \\
&= \{y \in \mathbb{R}^m : A^T y = 0\} = \mathcal{N}(A^T).
\end{aligned}$$

Math 330 homework 4

13. Let $A \in \mathbf{R}^{4\times 7}$.

(i) Find the matrix E such that AE corresponds to the result obtained after performing the column operation $c_2 \leftarrow c_2 - 3c_1$ on the matrix A.

$$\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix} \quad c_2 - 3c_1 \quad \underbrace{\begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{E}$$

(ii) Prove or disprove the claim that $\mathcal{N}(AE) = \mathcal{N}(A)$. If true explain why; if false provide a counterexample.

$$N(AE) = \{x : AEx = 0\} \qquad N(A) = \{x : Ax = 0\}$$

Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad AE = \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then $N(A) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

different spaces.

$N(AE) = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$