Math 330: Final Exam Version B

This exam is open book, open web-browser and open notes. Your desktop, web camera and audio are being remotely recorded by Proctorio. The difference between Quiz 1 and Quiz 2 is that Proctorio will be recording each of you individually rather than Zoom doing the same thing as part of a video conference call.

The following instructions have not changed; however, please read them anyway:

- Do not click on the I'm finished question in WebCampus until you have shown all your written work to web camera and clearly stated how many pages you will be scanning and turning in for later grading. There is no need to show any Julia work to the web camera because that has already been captured by the screen recorder.
- Starting now until after the semester ends on December 18 do not send email, text or any other type of message to anyone about questions appearing on this exam.
- If you have a paid membership to WolframAlpha, Chegg or other fee-based web service please log out of those services. As your screen will be recorded, also log out of bank accounts, personal email and online shopping sites.
- Make sure only one monitor is enabled on your computer. If you normally use two monitors, you will have to temporarily disconnect one of them during the exam as Proctorio can only record one screen at a time.
- Before starting the exam Proctorio will verify that your video, audio and desktop are properly being recording. You will then be asked to show your student identification and your desk and work environment to the web camera.
    - It is fine to have books, notes and blank paper on your desk.
    - It's better not to have any dogs, cats or or other people in the room as the presence of multiple faces might confuse the computerized face detector.
    - If your favorite cat jumps on your desk during the exam do not panic; simply remove it and continue working.
- Work each problem using pencil and paper or using a computer and Julia as appropriate. It is not allowed to send or receive email during the exam or upload questions to any web forum or homework service.
    - It is fine to use Google and similar web search engines.
    - It it fine to use the free WolframAlpha. Do not log in to a paid account.
    - You may use any non-interactive web resources during the exam.
    - You may read Stack Overflow or the Julia website, but do not post any questions to these or any similar forums during the exam.
    - It is fine to open a browser tab to read the text book; however, I would recommend downloading a `pdf` copy ahead of time and using that if needed.
- If you find an error in the test or are confused about a question, please explain carefully in writing what is wrong and include that with your best attempt at an answer.

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- When you are finished
  - Make sure your pages are consecutively numbered.
  - State how may pages you will be turning in using your microphone.
  - Show your work one page at a time to the web camera.
  - Hold each page steady for a count of 10 so the web camera can focus on it.
  - There is no need to show any Julia work to the web camera because that has already been captured by the screen recorder.
- After you have shown all your written work to the web camera, return to WebCampus, answer the I'm finished question and press submit to stop the recording.
- After you have ended Proctorio you must still upload high-resolution scans of all work to Final Upload on WebCampus along with any Julia programs and computer output used to arrive at your final answers.
  - Upload all your written work as a single `pdf` file.
  - For Julia you may upload a JupyterLab notebook as a separate `ipynb` file.
  - Do not change anything before uploading copies of your work for grading.
  - Please type a note in the comment panel for the Final Upload if you notice a mistake in your work that you want to let me know about.

Except for 11(iii) which is extra credit, please answer all of the following questions:

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this exam" followed by your signature as the answer to this question.

2. Consider the matrices A and B given by

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 0 & 1 \\ -6 & -1 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

(i) Explain in details why these matrices have the same determinant.

(ii) What is the common value of that determinant?

3. Consider the matrix

$$A = \begin{bmatrix} -3 & -6 \\ 6 & -1 \end{bmatrix}.$$

Write down the polynomial $p(\lambda) = \det(A - \lambda I)$.

4. Suppose $u, v \in \mathbf{R}^3$ are given by $u = (-3, 6, -1)$ and $v = (-7, 2, 2)$.

(i) Find $v + 3v$.

(ii) Find uv^T.

5. Let $x \in \mathbf{R}^{31}$ be such that $\|x\| = 9$. At most how many different entries of x could satisfy $|x_i| \geq 4$?

6. The matrix

$$A = \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$.

(i) Explain how to find eigenvectors corresponding to λ_1 and λ_2 by hand and then use a pencil and paper calculation to actually find them.

(ii) Check your answer with Julia using `eigvals` and `eigvecs` or plug your answer into the eigenvalue-eigenvector equation and verify your answer by hand.

7. Consider the over-determined system of linear equations $Ax = b$ where

$$A = \begin{bmatrix} -8 & 8 \\ 2 & 2 \\ 9 & 10 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix}.$$

For what value of x is $\|Ax - b\|$ minimized?

(i) Explain how to find x using the QR factorization.

(ii) Explain how to find x using the pseudo-inverse $A^\dagger$.

(iii) Use Julia or pencil and paper to actually find x.

8. Suppose A, B and C are matrices that satisfy $AB + C = I$.

(i) is it true or false that A must be square. If true explain why, if false provide a counter example.

(ii) Is it true or false that A and B^T must have the same dimensions. If true explain why, if false provide a counter example.

9. Consider the adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(i) How many vertices are in the corresponding directed graph?

(ii) Draw a picture of the directed graph.

(iii) Is there any matrix power A^k such that none of the entries of A^k are zero?

(iv) Starting at any given vertex is it possible to reach any other vertex in the graph?

10. Suppose p is a polynomial of degree $n-1$ or less given by $p(t) = c_1 + c_2 t + \cdots + c_n t^{n-1}$. Let $q(t) = (1 - t^2)p(t)$. Then $q(t) = d_1 + d_2 t + \cdots + d_m t^{m-1}$ for some vector d. Find a matrix D for which $d = Dc$. Give the entries of D and specify its dimensions.

11. Let $A \in \mathbf{R}^{5\times5}$ be a matrix and $v \in \mathbf{R}^5$ a vector such that v, Av, A^2v, A^3v, A^4v is a linearly independent sequence of vectors.

(i) Show the vectors $v, Av, A^2v, A^3v, A^4v, A^5v$ must be linearly dependent.

(ii) Let $B \in \mathbf{R}^{5\times6}$ be the matrix given by

$$B = \begin{bmatrix} v & Av & A^2v & \cdots & A^5v \end{bmatrix}$$

and $x \in \mathbf{R}^6$ a non-zero vector such that $Bx = 0$. Explain why $x_6 \neq 0$.

(iii) [Extra Credit] Define

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & -x_1/x_6 \\ 1 & 0 & 0 & 0 & -x_2/x_6 \\ 0 & 1 & 0 & 0 & -x_3/x_6 \\ 0 & 0 & 1 & 0 & -x_4/x_6 \\ 0 & 0 & 0 & 1 & -x_5/x_6 \end{bmatrix}.$$

Prove the eigenvalues of M and and eigenvalues of A are the same. Then numerically verify this using the following Julia commands

```
A=2*rand(5,5).-1
v=2*rand(5).-1
B=[v A*v A^2*v A^3*v A^4*v A^5*v]
P=[v A*v A^2*v A^3*v A^4*v]
y=inv(P)*A^5*v
x=[y; -1]
B*x
I5=diagm(ones(5))
M=[I5[:,2] I5[:,3] I5[:,4] I5[:,5] -x[1:5]/x[6]]
eigvals(A)
eigvals(M)
```

#1 I have worked independently on this exam
— Test Student

#2 i. The matrices have the same determinant because one can be transformed into the other using row operations of the form $r_i \leftarrow r_i - \alpha r_j$. In particular

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 0 & 1 \\ -6 & -1 & 5 \end{bmatrix} \qquad \begin{matrix} r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 + 3r_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 2 & -1 \end{bmatrix} \qquad r_3 \leftarrow r_3 + 2r_2$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix} = B$$

(ii) The common value of the determinant is

$$\det(A) = \det(B) = 2(-1)(5) = -10.$$

#3 The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & -6 \\ 6 & -1-\lambda \end{bmatrix} = (-3-\lambda)(-1-\lambda) + 36$$

$$= \lambda^2 + 4\lambda + 39.$$

#4 Suppose $u=(-3,6,-1)$ and $v=(-7,2,2)$.

(i) $v+3v = 4v = 4\begin{bmatrix}-7\\2\\2\end{bmatrix} = \begin{bmatrix}-28\\8\\8\end{bmatrix}$

(ii) $uv^T = \begin{bmatrix}-3\\6\\-1\end{bmatrix}\begin{bmatrix}-7,2,2\end{bmatrix} = \begin{bmatrix}21 & -6 & -6\\-42 & 12 & 12\\7 & -2 & -2\end{bmatrix}$

#5 Let $x \in \mathbb{R}^{31}$ be such that $\|x\|=9$. At most how many different entries of x could satisfy $|x_i| \geq 4$?

By Chebyshev inequality $k \leq \|x\|^2/a^2$ where $a=4$.

Since $9^2/4^2 = 5.0625$ then at most 5 entries of x could satisfy $|x_i| \geq 4$.

#6 The matrix $\begin{bmatrix}1 & -1\\-4 & 1\end{bmatrix}$ has eigenvalues $\lambda_1=-1$ and $\lambda_2=3$.

(1) Explain how to find the eigenvectors corresponding to λ_i by hand and then find them.

To find the eigenvectors by hand, plug $\lambda=\lambda_i$ into the eigenvalue-eigenvector equation

$$Ax=\lambda x \quad \text{or} \quad (A-\lambda I)x=0$$

and then solve for a non-zero x that satisfies this equation.

#6(ii) continues...

$\lambda_1 = -1$

$$\begin{bmatrix} 1-\lambda & -1 \\ -4 & 1-\lambda \end{bmatrix} x = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Therefore $x_1 = \frac{1}{2} x_2$ so that.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2\, x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} x_2$$

We conclude $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

$\lambda_2 = 3$

$$\begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Therefore $x_1 = -\frac{1}{2} x_2$ so that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2\, x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} x_2$$

We conclude that $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.

#7. Let

$$A = \begin{bmatrix} -8 & 8 \\ 2 & 2 \\ 9 & 10 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix}$$

For what value of x is $\|Ax-b\|$ minimized?

(i) Explain how to find x using QR factorization.

Suppose $A = QR$ where $Q \in \mathbb{R}^{3\times 2}$ and $R \in \mathbb{R}^{2\times 2}$.

Then $Rx = Q^T b$ and one can find x using back-substitution since R is upper triangular.

(ii) By definition $A^\dagger = (A^T A)^{-1} A^T$ and one can find x by computing $x = A^\dagger b$.

(iii) In Julia I typed

```
Q,R = qr(A)
Q = Matrix(Q)
x = R\Q'*b
```

and found that $x \approx \begin{bmatrix} 0.340129\ldots \\ 0.344024\ldots \end{bmatrix}$.

Please see the attached Jupyter-lab output for details of this computation.

#8. Suppose $AB + C = I$

(i) Is it true or false that A must be square?
This is false. For example

$$A \in \mathbb{R}^{2\times 3} \quad B \in \mathbb{R}^{3\times 2} \quad C \in \mathbb{R}^{2\times 2}$$

with $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Satisfy this equation. There are many other examples as well.

(ii) Is it true or false that A and B^T must have the same dimensions? This is true.

Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Then

for $\underset{m\times n}{A}\underset{p\times q}{B}$ to make sense

we must have that $n = p$.

Moreover if this product summed with C is the identity matrix, then since identity matrices are square it must be that $m = q$.

Thus
$$A \in \mathbb{R}^{q\times p} \text{ and so } A^T \in \mathbb{R}^{p\times q}$$

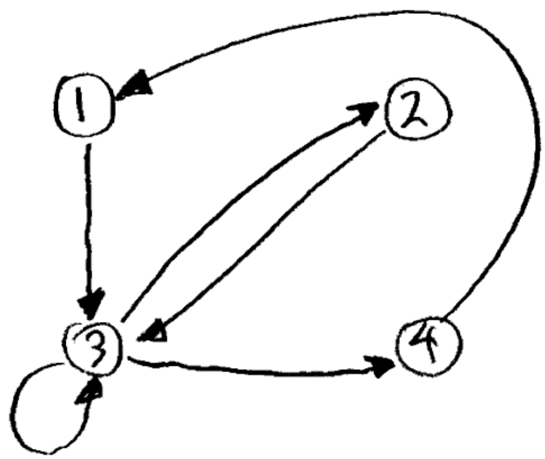
In particular A^T and B must have the same dimensions.

#9 Consider the adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(i) How many vertices are in the corresponding graph? Since $A \in \mathbb{R}^{4\times4}$ then there are 4 vertices in the graph.

(ii) Draw a picture of the directed graph



(iii) Is there any matrix power A^k such that none of the entries A^k are zero? <u>YES</u>

Computations in Julia attached yield that

$$A^4 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 1 \\ 4 & 4 & 7 & 2 \\ 2 & 2 & 4 & 1 \end{bmatrix}$$

Therefore $k=4$ works.

(iv) Since A^4 has no non-zero entries it is possible to reach any vertex from another by means of a path of length 4.

#10 Suppose p is a polynomial of degree $n-1$ or less given by $p(t) = c_1 + c_2 t + \cdots + c_n t^{n-1}$. Let $q(t) = (1-t^2)p(t)$. Then $q(t) = d_1 + d_2 t + \cdots + d_m t^{m-1}$ for some vector d. Find a matrix D for which $d = Dc$.

$$(1-t^2)(c_1 + c_2 t + \cdots + c_n t^{n-1})$$

$$= (c_1 + c_2 t + \cdots + c_n t^{n-1}) - (c_1 t^2 + c_2 t^3 + \cdots + c_n t^{n+1})$$

$$= c_1 + c_2 t + (c_3 - c_1) t^2 + \cdots + (c_n - c_{n-2}) t^{n-1} - c_{n-1} t^n - c_n t^{n+1}$$

Therefore $D \in \mathbb{R}^{n+2 \times n}$ and

$$D = n{+}2\left\{ \underbrace{\begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ -1 & 0 & 1 & \\ & -1 & 0 & \ddots \\ & & -1 & \ddots & \\ & & & \ddots & 1 \\ & 0 & & \ddots & 0 \\ & & & & -1 \end{bmatrix}}_{n} \right.$$

#11 Let $A \in \mathbb{R}^{5\times5}$ be a matrix and $v \in \mathbb{R}^5$ be a vector such that v, Av, A^2v, A^3v and A^4v are linearly independent.

(i) Show that $v, Av, A^2v, A^3v, A^4v, A^5$ must be linearly dependent.

Since $A \in \mathbb{R}^{5\times5}$ and $v \in \mathbb{R}^5$ then all these vectors are 5-vectors. By the dimension-independence inequality there can be no more than 5 linearly independent vectors in $\mathbb{R}^5$. Since we are considering 6 vectors, they must be linearly dependent.

(ii) Let $B \in \mathbb{R}^{5\times6}$ be the matrix with columns

$$B = \left[v \,|\, Av \,|\, \cdots \,|\, A^5 v \right]$$

and x a non-zero vector such that $Bx = 0$. Explain why $x_6 \neq 0$.

If $x_6 = 0$ then $Bx = 0$ would imply

$$x_1 v + x_2 Av + \cdots + x_5 A^4 v = 0$$

But then the independence of the original sequence of vectors would imply that $x_1 = x_2 = \cdots = x_5 = 0$, contradicting that x was non-zero.

Therefore, it must be that $x_6 \neq 0$.

#11(iii) [Extra Credit]

Define
$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & -x_1/x_6 \\ 1 & 0 & 0 & 0 & -x_2/x_6 \\ 0 & 1 & 0 & 0 & -x_3/x_6 \\ 0 & 0 & 1 & 0 & -x_4/x_6 \\ 0 & 0 & 0 & 1 & -x_5/x_6 \end{bmatrix}$$

Prove the eigenvalues of M and A are the same and then verify this with Julia.

Note that
$$x_1 v + x_2 A v + x_3 A^2 v + \cdots + x_6 A^5 v = 0$$

implies that
$$\begin{bmatrix} -x_1/x_6 \\ -x_2/x_6 \\ -x_3/x_6 \\ -x_4/x_6 \\ -x_5/x_6 \end{bmatrix} = P^{-1} A^5 v$$

where P is the matrix

$$P = \begin{bmatrix} v \,|\, Av \,|\, \cdots \,|\, A^4 v \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

For convenience let $y = P^{-1} A^5 v$. Note that P is invertible because the columns of P are linearly independent.

#11 (iii) continues...

Now, suppose z is an eigenvector of A with eigenvalue λ. Thus $Az = \lambda z$.

Define $y = P^{-1} z$. Therefore

$$\lambda P y = \lambda\left(y_1 v + y_2 Av + \cdots + y_5 A^4 v\right)$$
$$= \lambda y_1 v + \lambda y_2 Av + \cdots + \lambda y_5 A^4 v.$$

On the other hand

$$APy = A\left(y_1 v + y_2 Av + \cdots + y_5 A^4 v\right)$$
$$= y_1 Av + y_2 A^2 v + \cdots + y_5 A^5 v$$
$$= y_1 Av + y_2 A^2 v + \cdots + y_4 A^4 v + y_5 Py$$
$$= y_1 Av + y_2 A^2 v + \cdots + y_4 A^4 v$$
$$+ y_5\left(y_1 v + y_2 Av + \cdots + y_5 A^4 v\right)$$
$$= y_5 y_1 v + (y_1 + y_5 y_2) Av + \cdots + (y_4 + y_5 y_5) A^4 v$$

Since $APy = \lambda Py$ we conclude that

$$\lambda y_1 = y_5 y_1$$
$$\lambda y_2 = y_1 + y_5 y_2$$
$$\vdots$$
$$\lambda y_5 = y_4 + y_5 y_5$$

#11 (iii) Continues...

Claim that y is an eigenvector of M that also has eigenvalue λ.

Let e_i be the standard basis of $\mathbb{R}^5$. Then note that, by construction of M, that

$$e_2 = Me_1, \quad e_3 = M^2 e_1, \quad \dots, \quad e_5 = M^4 e_1.$$

Therefore

$$y = y_1 e_1 + y_2 e_2 + \dots + y_5 e_5$$
$$= y_1 e_1 + y_2 M e_1 + \dots + y_5 M^4 e_1$$

It follows that

$$My = y_1 M e_1 + y_2 M^2 e_1 + \dots + y_4 M^4 e_1 + y_5 M^5 e_1$$

By construction, again

$$M^5 e_1 = \begin{bmatrix} -x_1/x_0 \\ -x_2/x_0 \\ \vdots \\ -x_5/x_0 \end{bmatrix} = y$$

Therefore

$$My = y_1 M e_1 + y_2 M^2 e_1 + \dots + y_4 M^4 e_1 + y_5 y$$
$$= y_5 y_1 e_1 + (y_1 + y_5 y_2) M e_1 + \dots + (y_4 + y_5 y_5) M^4 e_1$$
$$= \lambda y_1 e_1 + \lambda y_2 M e_1 + \dots + \lambda y_5 M^4 e_1 = \lambda y.$$

#11 (iii) continues...

We have show that every eigenvalue of A is an eigenvalue of M. As this process is reversible it follows that every eigenvalue of M is also an eigenvalue of A.

This finishes the theoretical part of the extra credit. The Julia example is in the attached worksheet.

330finjulia

December 19, 2020

Math 330 Final Question 7 part (iii)

[35]:
```
using LinearAlgebra
```

[36]:
```
A=[-8 8; 2 2; 9 10]
```

[36]:
```
3×2 Array{Int64,2}:
 -8   8
  2   2
  9  10
```

[37]:
```
b=[0,-1,7]
```

[37]:
```
3-element Array{Int64,1}:
  0
 -1
  7
```

[38]:
```
Q,R=qr(A)
```

[38]:
```
LinearAlgebra.QRCompactWY{Float64,Array{Float64,2}}
Q factor:
3×3 LinearAlgebra.QRCompactWYQ{Float64,Array{Float64,2}}:
 -0.655386  -0.755185  -0.0128746
  0.163846  -0.125513  -0.978469
  0.737309  -0.643384   0.205993
R factor:
2×2 Array{Float64,2}:
 12.2066    2.4577
  0.0     -12.7263
```

[39]:
```
Q=Matrix(Q)
```

[39]:
```
3×2 Array{Float64,2}:
 -0.655386  -0.755185
  0.163846  -0.125513
  0.737309  -0.643384
```

[40]:
```
x=R\Q'*b
```

[40]:
```
2-element Array{Float64,1}:
 0.34012928891098965
 0.3440245317420852
```

Math 330 Final Question 9 part (iii)

[41]:
```
A=[0 0 0 1; 0 0 1 0; 1 1 1 0; 0 0 1 0]
```

[41]:
```
4×4 Array{Int64,2}:
 0  0  0  1
 0  0  1  0
 1  1  1  0
 0  0  1  0
```

[42]:
```
A^2
```

[42]:
```
4×4 Array{Int64,2}:
 0  0  1  0
 1  1  1  0
 1  1  2  1
 1  1  1  0
```

[43]:
```
A^3
```

[43]:
```
4×4 Array{Int64,2}:
 1  1  1  0
 1  1  2  1
 2  2  4  1
 1  1  2  1
```

[44]:
```
A^4
```

[44]:
```
4×4 Array{Int64,2}:
 1  1  2  1
 2  2  4  1
 4  4  7  2
 2  2  4  1
```

Math 330 Problem 11 part (iii) [Extra Credit]

[45]:
```
A=2*rand(5,5).-1
```

[45]:
```
5×5 Array{Float64,2}:
 -0.605066     0.06561   -0.862656  -0.576102   0.80677
  0.134474    -0.257231  -0.814115   0.13598    0.467734
 -0.00505576   0.497775  -0.807959  -0.506019   0.387067
```

```
  0.193933    -0.166722   0.552698  -0.368045  -0.348074
 -0.489903    -0.492042  -0.214049   0.62644   -0.675381
```

[46]:
```
v=2*rand(5).-1
```

[46]:
```
5-element Array{Float64,1}:
  0.47332468384901905
  0.48970387521107916
  0.5467043666017948
 -0.5513846349222349
  0.3673964551978002
```

[47]:
```
P=[v A*v A^2*v A^3*v A^4*v]
```

[47]:
```
5×5 Array{Float64,2}:
  0.473325  -0.111822  -1.32771    2.41578   -0.824037
  0.489704  -0.410531  -0.590096   1.46184   -0.616802
  0.546704   0.220873  -1.03631    0.81293    0.960118
 -0.551385   0.387365   0.438178  -1.32873    0.957079
  0.367396  -1.1834     1.25141    0.591935  -3.30894
```

[48]:
```
B=[P A^5*v]
```

[48]:
```
5×6 Array{Float64,2}:
  0.473325  -0.111822  -1.32771    2.41578   -0.824037  -3.59105
  0.489704  -0.410531  -0.590096   1.46184   -0.616802  -2.15136
  0.546704   0.220873  -1.03631    0.81293    0.960118  -2.84368
 -0.551385   0.387365   0.438178  -1.32873    0.957079   1.27319
  0.367396  -1.1834     1.25141    0.591935  -3.30894    3.33603
```

[49]:
```
y=inv(P)*A^5*v
```

[49]:
```
5-element Array{Float64,1}:
 -0.32833006229626893
 -1.7533770718244344
 -3.906745977803324
 -4.576119117193128
 -2.7136822361398294
```

[50]:
```
x=[y; -1]
```

[50]:
```
6-element Array{Float64,1}:
 -0.32833006229626893
 -1.7533770718244344
 -3.906745977803324
 -4.576119117193128
 -2.7136822361398294
```

```
 -1.0
```

[51]:
```
B*x
```

[51]:
```
5-element Array{Float64,1}:
 -2.042810365310288e-14
 -8.881784197001252e-16
 -3.5083047578154947e-14
 -6.661338147750939e-15
  7.949196856316121e-14
```

[52]:
```
I5=diagm(ones(5))
```

[52]:
```
5×5 Array{Float64,2}:
 1.0  0.0  0.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0
 0.0  0.0  0.0  0.0  1.0
```

[53]:
```
M=[I5[:,2:5] -x[1:5]/x[6]]
```

[53]:
```
5×5 Array{Float64,2}:
 0.0  0.0  0.0  0.0  -0.32833
 1.0  0.0  0.0  0.0  -1.75338
 0.0  1.0  0.0  0.0  -3.90675
 0.0  0.0  1.0  0.0  -4.57612
 0.0  0.0  0.0  1.0  -2.71368
```

[54]:
```
eigvals(A)
```

[54]:
```
5-element Array{Complex{Float64},1}:
 -0.6669710698775457 - 1.227860628347879im
 -0.6669710698775457 + 1.227860628347879im
 -0.4770290266418471 + 0.0im
 -0.4513555348714115 - 0.3857360862273473im
 -0.4513555348714115 + 0.3857360862273473im
```

[55]:
```
eigvals(M)
```

[55]:
```
5-element Array{Complex{Float64},1}:
 -0.6669710698775557 - 1.2278606283478608im
 -0.6669710698775557 + 1.2278606283478608im
 -0.4770290266418597 + 0.0im
 -0.4513555348714288 - 0.38573608622737854im
 -0.4513555348714288 + 0.38573608622737854im
```

```
[ ]:
```