Math 330 Linear Algebra Homework 11

10.3 *Matrix sizes.* Suppose A, B, and C are matrices that satisfy $A + BB^T = C$. Determine which of the following statements are necessarily true. (There may be more than one true statement.)

(a) A is square. ✓

(b) A and B have the same dimensions. Not necessarily

(c) A, B, and C have the same number of rows. ✓

(d) B is a tall matrix. Not necessarily

Since BB^T is square than both A and C need to be square.

If $B \in \mathbb{R}^{m \times n}$ then $\underset{m \times n \; n \times m}{BB^T} \in \mathbb{R}^{m \times m}$ so in particular $A \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{m \times m}$. This further implies both A, B and C have the same number of rows.

Note however that n does not need to be equal to m and so A and B may not have the same dimensions. Moreover, there is no reason that $m > n$ or $m < n$ so B could be either a tall matrix or a wide one.

10.6 *Product of rotation matrices.* Let A be the 2×2 matrix that corresponds to rotation by θ radians, defined in (7.1), and let B be the 2×2 matrix that corresponds to rotation by ω radians. Show that AB is also a rotation matrix, and give the angle by which it rotates vectors. Verify that $AB = BA$ in this case, and give a simple English explanation.

By (7.1) we have that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

similarly

$$B = \begin{bmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{bmatrix}.$$

Computing yields that

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos w - \sin\theta\sin w & -\cos\theta\sin w - \sin\theta\cos w \\ \sin\theta\cos w + \cos\theta\sin w & -\sin\theta\sin w + \cos\theta\cos w \end{bmatrix}.$$

Recall the angle addition formulas

$$\sin(a+b) = \sin a\cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a \sin b.$$

Applying these with $a = \theta$ and $b = w$ yields that

$$AB = \begin{bmatrix} \cos(\theta+w) & -\sin(\theta+w) \\ \sin(\theta+w) & \cos(\theta+w) \end{bmatrix}$$

which is the rotation matrix for the angle $\theta + w$.

10.6 continues...

also note that

$$BA = \begin{bmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{bmatrix} \begin{bmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{bmatrix}$$

is exactly the same, except with the roles of w and ϑ switched. Thus BA is also the rotation matrix for the angle $w + \vartheta = \vartheta + w$.

In English this means that rotating an object first by ϑ and then by w degrees is the same in two dimensions as rotating it first by w degrees and then by ϑ degrees.

Math 330 Linear Algebra Homework 11

10.11 *Trace of matrix-matrix product.* The sum of the diagonal entries of a square matrix is called the *trace* of the matrix, denoted $\mathbf{tr}(A)$.

(a) Suppose A and B are $m \times n$ matrices. Show that

$$\mathbf{tr}(A^T B) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}.$$

What is the complexity of calculating $\mathbf{tr}(A^T B)$?

(b) The number $\mathbf{tr}(A^T B)$ is sometimes referred to as the inner product of the matrices A and B. (This allows us to extend concepts like angle to matrices.) Show that $\mathbf{tr}(A^T B) = \mathbf{tr}(B^T A)$.

(c) Show that $\mathbf{tr}(A^T A) = \|A\|^2$. In other words, the square of the norm of a matrix is the trace of its Gram matrix.

(d) Show that $\mathbf{tr}(A^T B) = \mathbf{tr}(BA^T)$, even though in general $A^T B$ and BA^T can have different dimensions, and even when they have the same dimensions, they need not be equal.

(a) By definition $C = \underset{n\times m}{A^T} \underset{m\times n}{B}$ means

$$C_{ij} = \sum_{k=1}^{m} (A^T)_{ik} B_{kj} = \sum_{k=1}^{m} A_{ki} B_{kj}$$

and the trace is

$$tr(A^T B) = tr\, C = \sum_{i=1}^{n} C_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{m} A_{ki} B_{ki}$$

The complexity of calculating in $O(mn)$.

(b) Let $D = \underset{n\times m}{B^T} \underset{m\times n}{A}$. Then

$$D_{ij} = \sum_{k=1}^{m} (B^T)_{ik} A_{kj} = \sum_{k=1}^{m} B_{ki} A_{kj}$$

and

$$tr(B^T A) = tr(D) = \sum_{i=1}^{n} D_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{m} B_{ki} A_{ki}.$$

which is the same as $tr(A^T B)$.

10.11 continues...

(c) By definition $\|A\|^2 = \sum_{i,j} |A_{ij}|^2$. Now

$$tr\, A^T A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} A_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2 = \|A\|^2.$$

(d) Let $E = \underset{m\times n}{B} \underset{n\times m}{A^T}$. Then

$$E_{ij} = \sum_{k=1}^{n} B_{ik} (A^T)_{kj} = \sum_{k=1}^{n} B_{ik} A_{jk}$$

and the trace is

$$tr(BA^T) = tr(E) = \sum_{i=1}^{m} E_{ii} = \sum_{i=1}^{m} \sum_{k=1}^{n} B_{ik} A_{ik}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{m} B_{ik} A_{ik} = tr(A^T B).$$

10.35 *Orthogonal matrices.* Let U and V be two orthogonal $n \times n$ matrices. Show that the matrix UV and the $(2n) \times (2n)$ matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} U & U \\ V & -V \end{bmatrix}$$

are orthogonal.

Orthogonal means that the matrix is square and that $U^TU = I$ and $V^TV = I$.

Now

$$A^T = \frac{1}{\sqrt{2}} \begin{bmatrix} U^T & V^T \\ U^T & -V^T \end{bmatrix}$$

thus

$$A^TA = \frac{1}{2} \begin{bmatrix} U^T & V^T \\ U^T & -V^T \end{bmatrix} \begin{bmatrix} U & U \\ V & -V \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} U^TU + V^TV & U^TU - V^TV \\ U^TU - V^TV & U^TU + V^TV \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2I & 0 \\ 0 & 2I \end{bmatrix} = I$$

Therefore the matrix A is orthogonal.

Math 330 Linear Algebra Homework 11

10.3 *State feedback control.* Consider a time-invariant linear dynamical system with n-vector state x_t and m-vector input u_t, with dynamics

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots.$$

The entries of the state often represent deviations of n quantities from their desired values, so $x_t \approx 0$ is a goal in operation of the system. The entries of the input u_t are deviations from the standard or nominal values. For example, in an aircraft model, the states might be the deviation from the desired altitude, climb rate, speed, and angle of attack; the input u_t represents changes in the control surface angles or engine thrust from their normal values.

In *state feedback control*, the states are measured and the input is a linear function of the state, $u_t = Kx_t$. The $m \times n$ matrix K is called the *state feedback gain matrix*. The state feedback gain matrix is very carefully designed, using several methods. State feedback control is very widely used in many application areas (including, for example, control of airplanes).

(a) *Open and closed-loop dynamical system.* With $u_t = 0$, the system satisfies $x_{t+1} = Ax_t$ for $t = 1, 2, \ldots$, which is called the *open-loop dynamics*. When $u_t = Kx_t$, the system dynamics can be expressed as $x_{t+1} = \tilde{A}x_t$, for $t = 1, 2, \ldots$, where the $n \times n$ matrix $\tilde{A}$ is the *closed-loop dynamics matrix*. Find an expression for $\tilde{A}$ in terms of A, B, and K.

(b) *Aircraft control.* The longitudinal dynamics of a 747 flying at 40000 ft at Mach 0.81 is given by

$$A = \begin{bmatrix} .99 & .03 & -.02 & -.32 \\ .01 & .47 & 4.7 & .00 \\ .02 & -.06 & .40 & -.00 \\ .01 & -.04 & .72 & .99 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.01 & 0.99 \\ -3.44 & 1.66 \\ -0.83 & 0.44 \\ -0.47 & 0.25 \end{bmatrix},$$

where the sampling time is one second. (The state and control variables are described in more detail in the lecture on control.) We will use the state feedback matrix

$$K = \begin{bmatrix} -.038 & .021 & .319 & -.270 \\ -.061 & -.004 & -.120 & .007 \end{bmatrix}.$$

(The matrices A, B, and K can be found in `747_cruise_dyn_data.jl`, so you don't have to type them in.) Plot the open-loop and closed-loop state trajectories from several nonzero initial states, such as $x_1 = (1, 0, 0, 0)$, or ones that are randomly generated, from $t = 1$ to $t = 100$ (say). (In other words, plot $(x_t)_i$ versus t, for $i = 1, 2, 3, 4$.) Would you rather be a passenger in the plane with the state feedback control turned off (*i.e.*, open-loop) or on (*i.e.*, closed-loop)?

(a) Since $x_{t+1} = Ax_t + Bu_t$, then taking $u_t = Kx_t$ and substituting it in yields

$$x_{t+1} = Ax_t + B(Kx_t) = (A+BK)x_t$$

Therefore $\tilde{A} = A + BK$.

hw11p103b

December 2, 2020

Homework 11 Question 10.3b

[1]:
```
A=[.99 .03 -.02 -.32;
   .01 .47  4.7  .0;
   .02 -.06 .40  .0;
   .01 -.04 .72  .99]
```

[1]:
```
4×4 Array{Float64,2}:
 0.99   0.03  -0.02  -0.32
 0.01   0.47   4.7    0.0
 0.02  -0.06   0.4    0.0
 0.01  -0.04   0.72   0.99
```

[2]:
```
B=[0.01 0.99; -3.44 1.66;
  -0.83 0.44; -0.47 0.25]
```

[2]:
```
4×2 Array{Float64,2}:
  0.01  0.99
 -3.44  1.66
 -0.83  0.44
 -0.47  0.25
```

[3]:
```
K=[-.038 .021 .319 -.270;
   -.061 -.004 -.12 .007]
```

[3]:
```
2×4 Array{Float64,2}:
 -0.038   0.021   0.319  -0.27
 -0.061  -0.004  -0.12    0.007
```

[4]:
```
Aclosed=A+B*K
```

[4]:
```
4×4 Array{Float64,2}:
 0.92923   0.02625  -0.13561  -0.31577
 0.03946   0.39112   3.40344   0.94042
 0.0247   -0.07919   0.08243   0.22718
 0.01261  -0.05087   0.54007   1.11865
```

Plot the open-loop trajectories for $x_1 = (1, 0, 0, 0)$ from $t = 1$ to $t = 100$.

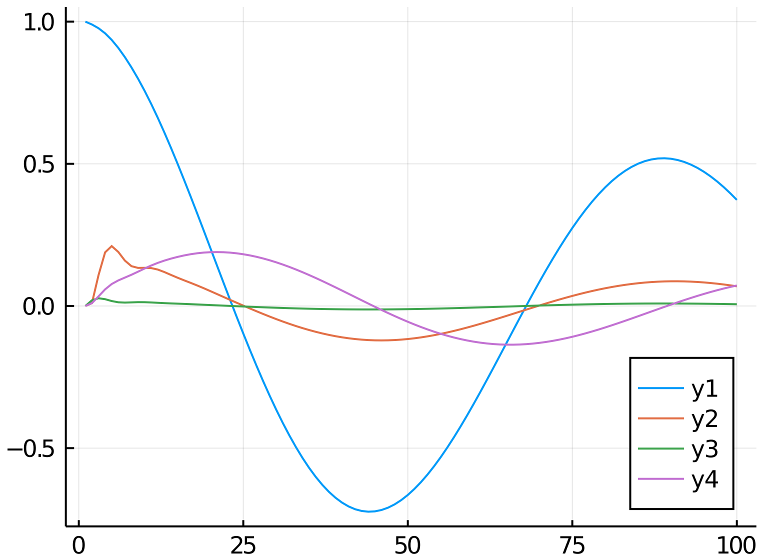
[25]:
```
Xopen=zeros(4,100);
```

[26]:
```
Xopen[:,1]=[1,0,0,0]
for t=1:99
    Xopen[:,t+1]=A*Xopen[:,t]
end
```

[27]:
```
using Plots
```

[28]:
```
plot(Xopen',size=[400,300],legend=:bottomright)
```

[28]:

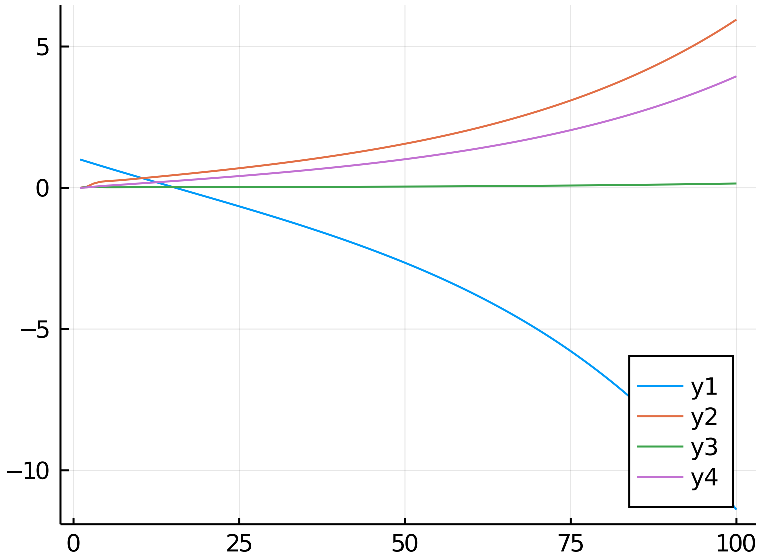


[9]:
```
Xclosed=zeros(4,100);
```

[29]:
```
Xclosed[:,1]=[1,0,0,0]
for t=1:99
    Xclosed[:,t+1]=Aclosed*Xclosed[:,t]
end
```

[30]:
```
plot(Xclosed',size=[400,300],legend=:bottomright)
```

[30]:



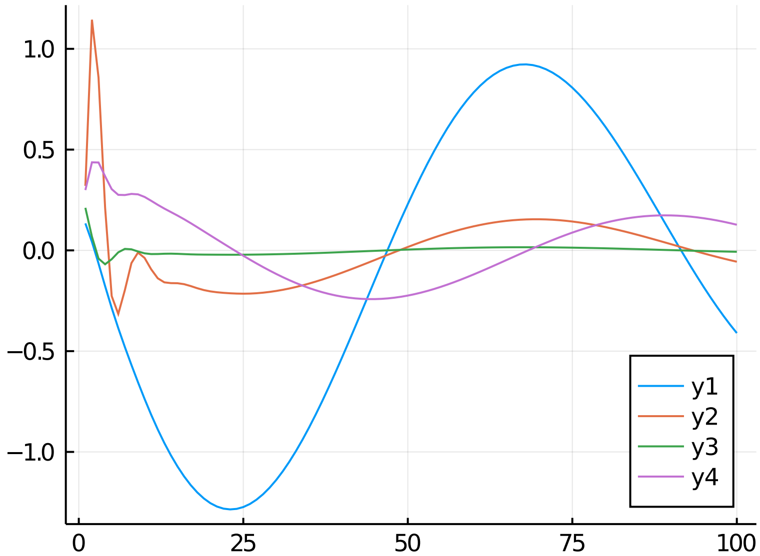
Note that the trajectories from the open-loop system oscillate up and down while the trajectories from the closed-loop system do not oscillate. While the oscillations might be exciting, I would generally prefer to be a passenger in a plane that doesn't oscillate up and down.

Here is another example starting from a random initial states.

[31]:
```
randstate=rand(4)
Xopen[:,1]=randstate
for t=1:99
    Xopen[:,t+1]=A*Xopen[:,t]
end
Xclosed[:,1]=randstate
for t=1:99
    Xclosed[:,t+1]=Aclosed*Xclosed[:,t]
end
```

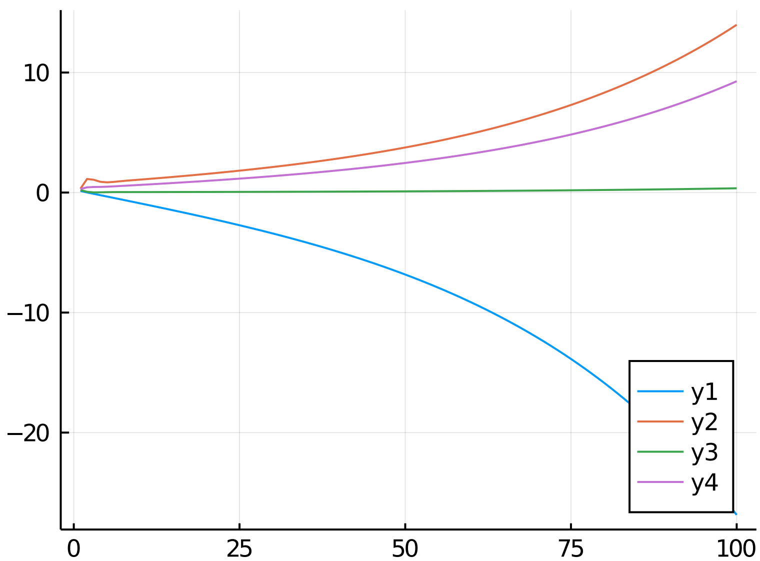
[32]:
```
plot(Xopen',size=[400,300],legend=:bottomright)
```

[32]:



[33]:
```
plot(Xclosed',size=[400,300],legend=:bottomright)
```

[33]:



Again the open-loop trajectories oscillate while the closed-loop trajectories do not oscillate.

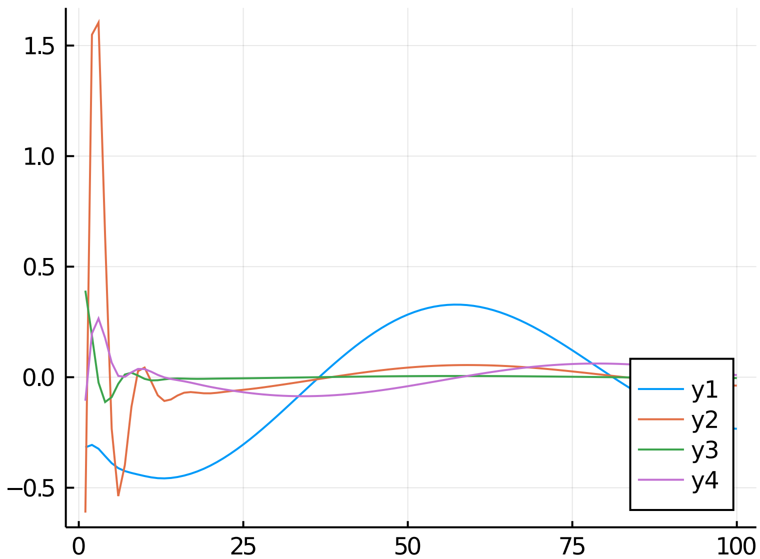
One more example, for completeness.

[38]:
```
randstate=2*rand(4).-1
Xopen[:,1]=randstate
```

```
for t=1:99
    Xopen[:,t+1]=A*Xopen[:,t]
end
Xclosed[:,1]=randstate
for t=1:99
    Xclosed[:,t+1]=Aclosed*Xclosed[:,t]
end
```

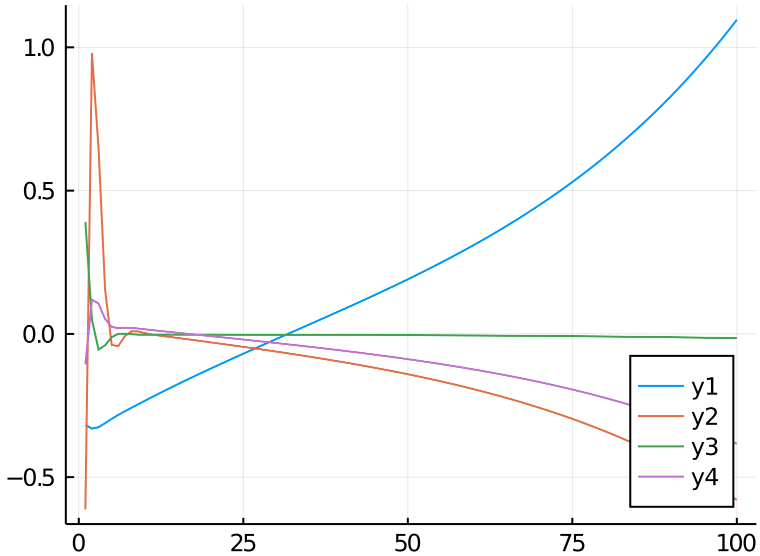
[39]:
```
plot(Xopen',size=[400,300],legend=:bottomright)
```

[39]:



[40]:
```
plot(Xclosed',size=[400,300],legend=:bottomright)
```

[40]:



While the red curve oscillates a bit even for the closed-loop system, the amount of oscillations are less than for the open-loop system. Thus, qualitatively similar results are obtained in each of the cases.

In particular, as already explained, I would rather be a passenger in with the state feedback control turned on.

[ ]:

```

```

Math 330 Linear Algebra Homework 11

10.7 Suppose $F^T G = 0$, where F and G are $n \times k$ matrices. Determine whether each of the following statements must always be true, or can be false. 'Must be true' means the statement holds for any $n \times k$ matrices F and G that satisfy $F^T G = 0$, without any further assumptions; 'can be false' means that there are $n \times k$ matrices F and G that satisfy $F^T G = 0$, but the statement does not hold.

(a) Either $F = 0$ or $G = 0$.

(b) The columns of F are orthonormal.

(c) Each column of F is orthogonal to each column of G.

(d) The matrices F and G are square or tall, *i.e.*, $n \geq k$.

(e) The columns of F are linearly dependent.

(a) Can be false. For example $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then

$F^T G = [0 \ 1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$ but neither F nor G are zero themselves.

(b) Can be false. F could be the zero matrix without any further assumptions and the columns of any zero matrix are not orthonormal.

(c) True, by definition of the inner product form of the matrix-matrix product if

$$F = [f_1 | f_2 | \cdots | f_k] \quad \text{and} \quad G = [g_1 | g_2 | \cdots | g_k]$$

Then

$$F^T G = \begin{bmatrix} f_1^T g_1 & \cdots & f_1^T g_k \\ \vdots & \ddots & \vdots \\ f_k^T g_1 & \cdots & f_k^T g_k \end{bmatrix} = 0$$

implies $f_i^T g_j = 0$ for all $i, j = 1, \ldots, k$.

#10.7 continues...

(d) Could be false. If one is the zero matrix then $F^T G = 0$ no matter whether $n < k$, $n = k$ or $n > k$.

(e) Could be false. In the first example for (a) the matrix $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has a single linearly independent column.