Math 330 Linear Algebra Homework 14

2. Prove that the following identity holds by expressing the left–hand side as the sum of 8 determinants:

$$\begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.$$

Since the determinant is a linear function in each of the rows, then

$$\begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix}$$

Continuing we obtain

$$\begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix}$$

Where

$$\begin{vmatrix} a & b & c \\ x & y & z \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} \to 0$$

and

$$\begin{vmatrix} a & b & c \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ u & v & w \\ u & v & w \end{vmatrix} \to 0 + \begin{vmatrix} a & b & c \\ u & v & w \\ a & b & c \end{vmatrix} \to 0$$

These are four of the eight determinants.

#2 continues...

To obtain the other four determinants continue to expand the remaining terms as

$$\begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \underbrace{\begin{vmatrix} x & y & z \\ x & y & z \\ u+a & v+b & w+c \end{vmatrix}}_{\text{(There were two term here)}} + \begin{vmatrix} x & y & z \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix}$$

where

$$\begin{vmatrix} x & y & z \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} x & y & z \\ u & v & w \\ u & v & w \end{vmatrix} + \begin{vmatrix} x & y & z \\ u & v & w \\ a & b & c \end{vmatrix}$$

Consequently

$$\begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} x & y & z \\ u & v & w \\ a & b & c \end{vmatrix}$$

The second term can be transformed swapping rows

$$\begin{vmatrix} x & y & z \\ u & v & w \\ a & b & c \end{vmatrix} = - \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}$$

Therefore

$$\begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.$$

4. Evaluate the following determinants:

(a) $\begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}$.

(a) Use linearity in the first column to obtain

$$\begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix} = 2\begin{vmatrix} 123 & 427 & 327 \\ 507 & 543 & 443 \\ -171 & 721 & 621 \end{vmatrix}$$

Now replace $r_2 \leftarrow r_2 + 3r_3$ so

$$= 2\begin{vmatrix} 123 & 427 & 327 \\ -6 & 2706 & 2306 \\ -171 & 721 & 621 \end{vmatrix}$$

Swap row as

$$= -2\begin{vmatrix} -6 & 2706 & 2306 \\ 123 & 427 & 327 \\ -171 & 721 & 621 \end{vmatrix}$$

Use linearity in the first row to obtain

$$= -4\begin{vmatrix} -3 & 1353 & 1153 \\ 123 & 427 & 327 \\ -171 & 721 & 621 \end{vmatrix}$$

Now eliminate by replacing $r_2 \leftarrow r_2 + 41r_1$

and $r_3 \leftarrow r_3 - 57r_1$

#4 continues...

$$\ldots = -4 \begin{vmatrix} -3 & 1353 & 1153 \\ 0 & 55900 & 47600 \\ 0 & -76400 & -65100 \end{vmatrix}$$

Now linearity in the second and third rows implies

$$= -40000 \begin{vmatrix} -3 & 1353 & 1153 \\ 0 & 559 & 476 \\ 0 & -764 & -651 \end{vmatrix}$$

At this point expanding the determinant along the first column yields

$$= (-40000)(-3) \begin{vmatrix} 559 & 476 \\ -764 & -651 \end{vmatrix}$$

Now taking $r_2 \leftarrow r_2 + r_1$ yields that

$$\begin{vmatrix} 559 & 476 \\ -764 & -651 \end{vmatrix} = \begin{vmatrix} 559 & 476 \\ -205 & -175 \end{vmatrix}$$

Then linearity in the second row gives

$$= 5 \begin{vmatrix} 559 & 476 \\ -41 & -35 \end{vmatrix}$$

and $r_1 \leftarrow r_1 + 14 r_2$ yields

$$= 5 \begin{vmatrix} -15 & -14 \\ -41 & -35 \end{vmatrix} = 5(15 \cdot 35 - 14 \cdot 41) = -245.$$

#4 continues,...

Putting everything together, we have

$$\begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix} = (-40000)(-3)(-245)$$

$$= -29{,}400{,}000.$$

#4 (b) By replacing rows as

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}$$

$r_2 \leftarrow r_2 + 2r_1$
$r_3 \leftarrow r_3 - 3r_1$
$r_4 \leftarrow r_4 - 4r_1$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 11 \\ 0 & -10 & -10 & -10 \\ 0 & -5 & -14 & -17 \end{vmatrix}$$

$r_3 \leftarrow r_3 + 2r_2$
$r_4 = r_4 + r_2$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 11 \\ 0 & 0 & -6 & 12 \\ 0 & 0 & -12 & -6 \end{vmatrix}$$

$r_4 \leftarrow r_4 - 2r_3$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 11 \\ 0 & 0 & -6 & 12 \\ 0 & 0 & 0 & -30 \end{vmatrix} = 5 \cdot 6 \cdot 30 = 900.$$

5. Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$$

by first computing the adjoint matrix.

$$C_{11} = M_{11}(A) = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -3-8 = -11$$

$$C_{12} = -M_{12}(A) = -\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix} = -(-9-20) = 29$$

$$C_{13} = M_{13}(A) = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6-5 = 1$$

$$C_{21} = -M_{21}(A) = -\begin{vmatrix} 0 & -2 \\ 2 & -3 \end{vmatrix} = -4$$

$$C_{22} = M_{22}(A) = \begin{vmatrix} 1 & -2 \\ 5 & -3 \end{vmatrix} = -3+10 = 7$$

$$C_{23} = -M_{23}(A) = -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2$$

$$C_{31} = M_{31}(A) = \begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} = 2$$

$$C_{32} = -M_{32}(A) = -\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = -(4+6) = -10$$

$$C_{33} = M_{33}(A) = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -11 & -4 & 2 \\ 29 & 7 & -10 \\ 1 & -2 & 1 \end{bmatrix}$$

#5 continues...

$$\det(A) = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}$$

$$= -11 - 2 \cdot 1 = -13$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{11}{13} & \frac{4}{13} & \frac{-2}{13} \\ \frac{-29}{13} & \frac{-7}{13} & \frac{10}{13} \\ \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \end{bmatrix}.$$

8. Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}.$$

Find the values of k for which $\det A = 0$ and hence, or otherwise, determine the value of k for which the following system has more than one solution:

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2. \end{aligned}$$

Solve the system for this value of k and determine the solution for which $x^2 + y^2 + z^2$ has least value.

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & k \\ -1 & k & 3 \end{vmatrix} \qquad \begin{matrix} r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 + r_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & k-1 \\ 0 & k+2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & k-1 \\ k+2 & 4 \end{vmatrix} = 4 - (k-1)(k+2)$$

$$= 4 - (k^2 + k - 2) = -(k^2 + k - 6)$$

Factoring

$$= -(k+3)(k-2) = 0$$

Therefore $k = -3$ or $k = 2$, then $\det A = 0$ and the system may have more than one solution.

#8 continues...

Of course the system may have no solutions, so plug the values of k in and check.

Case $k=-3$. Then

$$\begin{cases} x+y-z=1 \\ 2x+3y-3z=3 \\ x-3y+3z=2 \end{cases}$$

adding these two equations yields

$$3x=5 \quad \text{so} \quad x=\frac{5}{3}$$

Plugging this back in gives

$$\frac{5}{3}-y-z=1$$
$$\frac{10}{3}+3y-3z=3$$
$$\frac{5}{3}-3y+3z=2$$

or that

$$y-z=-\frac{2}{3}$$
$$3y-3z=-\frac{1}{3}$$
$$-3y+3z=\frac{1}{3}$$

these are the same ...

#8 continues...

Therefore we solve

$$3\left(y - z = -\frac{2}{3}\right)$$

$$3y - 3z = -\frac{1}{3}$$

Multiplying by the to equation by 3 and then subtracting yields

$$0 = -\frac{7}{3}$$

which implies there are no solutions when $k=-3$.

<u>Case</u> $k=2$. Then

$$x + y - z = 1$$
$$2x + 3y + 2z = 3$$
$$x + 2y + 3z = 2$$

Subtracting twice the top equation from the second and also subtracting it from the third yields

$$\left.\begin{array}{l} y + 4z = 1 \\ y + 4z = 1 \end{array}\right\} \text{The same equations}$$

We therefore consider the reduced system

#8 continues...

$$x + y - z = 1$$
$$y + 4z = 1$$

Solving for y in terms of z in the second equation gives

$$y = 1 - 4z$$

Plugging this into the first the obtains

$$x + (1 - 4z) - z = 1$$

$$x = 5z$$

Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5z \\ 1-4z \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} z .$$

We now choose z in order to minimize the term $x^2 + y^2 + z^2$.

This is, infact, a least squares problem in only one variable.

#8 continues...

In particular, we wish to find the least squares solution to

$$\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} z = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

This can be done by using the pseudo-inverse

$$\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}^{\#} = \left(\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}^T \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}^T$$

$$= \frac{1}{42} \begin{bmatrix} 5 & -4 & 1 \end{bmatrix}$$

Consequently, the minimal solution is

$$z = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}^{\#} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{42} \begin{bmatrix} 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \frac{2}{21}.$$

and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \frac{2}{21} = \begin{bmatrix} 10/21 \\ 13/21 \\ 2/21 \end{bmatrix}.$$