Math 330 Linear Algebra Homework 8

6.2 *Matrix notation.* Suppose the block matrix

$$\begin{bmatrix} A & I \\ I & C \end{bmatrix}$$

makes sense, where A is a $p \times q$ matrix. What are the dimensions of C?

Since the identity matrices are square we have the following

$$\begin{array}{cc} & \begin{array}{cc} q & p \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left[\begin{array}{c|c} A & I_p \\ \hline I_q & C \end{array}\right] \end{array}$$

Since the identity to the right of A must have the same number of rows as A and the identity below the same number of columns.

It follows that C is a $q \times p$ matrix.

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6.3 *Block matrix.* Assuming the matrix

$$K = \begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix}$$

makes sense, which of the following statements must be true? ('Must be true' means that it follows with no additional assumptions.)

(a) K is square. T

(b) A is square or wide. F

(c) K is symmetric, *i.e.*, $K^T = K$. T

(d) The identity and zero submatrices in K have the same dimensions. F

(e) The zero submatrix is square. T

Since the identity matrix is square, then A must have the same number of columns as the number of rows in A^T. This is always true, so this yields no additional condition on A. On the other hand this same argument run the other way implies that 0 is square, which wasn't known a priori. Note that since A need not be square that I and 0 need not be the same size. Having said this.

(a) K is square is true

(b) A is square or wide is false because it could have been tall.

(c) $K^T =$

(c) K is symmetric is true since

$$K^T = \begin{bmatrix} 1 & A^T \\ A & 0 \end{bmatrix}^T = \begin{bmatrix} 1^T & A^T \\ (A^T)^T & 0^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & A^T \\ A & 0 \end{bmatrix} = K.$$

(d) The identity need not have the same dimension as the zero submatrix. Thus the answer is false.

(e) The zero submatrix is square is true.

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6.6 *Matrix-vector multiplication.* For each of the following matrices, describe in words how x and $y = Ax$ are related. In each case x and y are n-vectors, with $n = 3k$.

(a) $A = \begin{bmatrix} 0 & 0 & I_k \\ 0 & I_k & 0 \\ I_k & 0 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}$, where E is the $k \times k$ matrix with all entries $1/k$.

Let $x \in \mathbb{R}^{3k}$ be written is stacked form as $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a, b, c \in \mathbb{R}^k$.

(a)

$$y = \begin{bmatrix} 0 & 0 & I_k \\ 0 & I_k & 0 \\ I_k & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

Thus y is the same as x, except that the first third and last thirds of the vector have been interchanged.

(b)

Note that $(Ea)_j = \sum_{i=1}^{k} \frac{a_i}{k} = \frac{1}{k} \sum_{i=1}^{k} a_i$ is the average of the entries in a for $j = 1, \ldots, k$.

Thus

$$y = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} Ea \\ Eb \\ Ec \end{bmatrix}$$

is a new vector in which every entry in the first third of the vector x have been replaced by their averge value, the middle replaced by the average of the middle and the last third replaced by the averge value of the last third.

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6.12 *Skew-symmetric matrices.* An $n \times n$ matrix A is called *skew-symmetric* if $A^T = -A$, *i.e.*, its transpose is its negative. (A symmetric matrix satisfies $A^T = A$.)

(a) Find all 2×2 skew-symmetric matrices.

(b) Explain why the diagonal entries of a skew-symmetric matrix must be zero.

(c) Show that for a skew-symmetric matrix A, and any n-vector x, $(Ax) \perp x$. This means that Ax and x are orthogonal. *Hint.* First show that for any $n \times n$ matrix A and n-vector x, $x^T(Ax) = \sum_{i,j=1}^n A_{ij}x_i x_j$.

(d) Now suppose A is any matrix for which $(Ax) \perp x$ for any n-vector x. Show that A must be skew-symmetric. *Hint.* You might find the formula

$$(e_i + e_j)^T(A(e_i + e_j)) = A_{ii} + A_{jj} + A_{ij} + A_{ji},$$

valid for any $n \times n$ matrix A, useful. For $i = j$, this reduces to $e_i^T(Ae_i) = A_{ii}$.

(a) Suppose $A^T = -A$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

implies

$a = -a$, $c = -b$, $b = -c$ and $d = -d$.

The first and last equation yield that $a = 0$ and $d = -d$ while the other two equations are the same. Thus the set of all 2×2 skew-symmetric matrices is given by

$$\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\}.$$

(b) The diagonal entries must be zero because

$$A^T = -A$$

implies $A_{ji} = -A_{ij}$ and when $i=j$ this means

$$A_{ii} = -A_{ii}$$

so that the diagonal entries $A_{ii} = 0$.

(c) The definition of matrix vector product implies $c = Ax$ satisfies

$$c_i = \sum_{j=1}^{n} A_{ij} x_j$$

The inner product with c and x is

$$x^T c = \sum_{i=1}^{n} x_i c_i = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

Switching the order of summation and using the fact that $A_{ji} = -A_{ij}$ we have

$$x^T c = \sum_{j=1}^{n} \sum_{i=1}^{n} A_{ij} x_i x_j = -\sum_{j=1}^{n} \sum_{i=1}^{n} A_{ji} x_j x_i$$

Now, relabeling we see that

$$x^T c = -\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij} x_i x_j = -x^T c$$

Consequently $x^T c = 0$, or in other-words

$$x^T A x = 0 \quad \text{when} \quad A^T = -A.$$

(d) Suppose $x^T A x = 0$ for every x. Taking $x = e_i$ we obtain that

$$e_i^T A e_i = A_{ii} = 0 \quad \text{for every } i.$$

Now

$$(e_i + e_j)^T A (e_i + e_j) =$$

$$= \cancel{e_i^T A e_i}^{\,0} + e_i^T A e_j + e_j^T A e_i + \cancel{e_j^T A e_j}^{\,0}$$

$$= 0 + A_{ij} + A_{ji} + 0 = 0$$

Therefore $A_{ji} = -A_{ij}$. This implies that $A^T = -A$ or that A is skew symmetric.

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6.13 *Polynomial differentiation.* Suppose p is a polynomial of degree $n-1$ or less, given by $p(t) = c_1 + c_2 t + \cdots + c_n t^{n-1}$. Its derivative (with respect to t) $p'(t)$ is a polynomial of degree $n-2$ or less, given by $p'(t) = d_1 + d_2 t + \cdots + d_{n-1} t^{n-2}$. Find a matrix D for which $d = Dc$. (Give the entries of D, and be sure to specify its dimensions.)

Note that $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^{n-1}$, therefore we have that $D \in \mathbb{R}^{n-1 \times n}$, i.e. that D is a $n-1 \times n$ matrix in order for the dimensions to be compatible.

Since the power rule implies

$$p'(t) = c_2 + 2c_3 + \cdots + (n-1)c_n$$

It follows that

$$d_1 = c_2, \quad d_2 = 2c_3, \ \ldots, \ d_{n-1} = (n-1)c_n.$$

Consequently the matrix D is

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 0 & 2 & 0 & & 0 & & 0 \\ 0 & 0 & 0 & 3 & \cdots & & & 0 \\ & & & & \ddots & & & \vdots \\ & 0 & & & & 0 & n-2 & 0 \\ & & & & & 0 & 0 & n-1 \end{bmatrix}$$