Find the singular value decomposition of A

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

note in this A is not square, so obviously it's not invertible...

$$B = A^T A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Note

- B is invertible (good luck)
- B is diagonal (very good luck)

By the spectral theorem B has an orthonormal eigenbasis...

$Be_1 = 2e_1$ $\qquad$ $Be_2 = 3e_2$

Eigenbasis $\{e_2, e_1\}$ so the eigenvalues $\lambda_1 \geq \lambda_2$

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{\begin{matrix} x_1 = e_2 & \lambda_1 = 3 \\ x_2 = e_1 & \lambda_2 = 2 \end{matrix}}$$

Find y's and then v's $\qquad y_i = Ax_i$

$$y_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} x_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} e_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} x_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} e_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Thus,

$$v_1 = \frac{y_1}{\|y_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \frac{y_2}{\|y_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\sqrt{\lambda_1}$ $\qquad$ $\sqrt{\lambda_2}$

if $\|y_i\| \neq \sqrt{\lambda_i}$ then we made a mistake...

$$V = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

looking for

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = V \Sigma U^T$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \qquad U = \begin{bmatrix} x_1 \mid x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Two options (Reduced SVD)

orthogonal matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

only a matrix with orthonormal columns.

diagonal matrix

use Gram-Schmidt to extend in general...

Other option: extend V with one more vector so it's orthogonal and then extend
(Full SVD)
Σ so you can still multiply stuff together...

Since all I want is the missing vector in three space cross product will also work...

$$y_1 \times y_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \det \begin{bmatrix} i & 1 & 1 \\ j & 1 & 0 \\ k & 1 & -1 \end{bmatrix}$$

$$= i(-1) - j(-1-1) + k(-1) = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$v_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

Full SVD:

extend Σ so can mult

orthogonal matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{2\times 2}$$

orthogonal matrix

diagonal but not square...