Math 330: Final Exam Version A Sample Final

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Write down the augmented matrix $[\,A\,|\,b\,]$ corresponding to the system of linear equations given by

not a typo... the columns don't line up...

$$\begin{cases} 2x_1 - 2x_2 + x_4 = 5 \\ x_2 - 5x_3 - 5x_4 = 2 \\ -2x_1 + 3x_2 - 5x_4 = -7 \end{cases}$$

but *do not* solve these equations.

From exam 1

note use $\det(AB) = (\det A)(\det B)$ to save time and reduce mistakes...

From Exam 2

3. Find $\det(A)$, $\det(B)$ and $\det(AB)$ where

$$A = \begin{bmatrix} 1 & 2 & 17 \\ 0 & -2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_1 \leftrightarrow r_2 \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_2 \leftrightarrow r_3 \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_1 \leftarrow 2r_1$
$r_2 \leftarrow 2r_2$
$r_3 \leftarrow 2r_3$

from Exam 2

4. Consider the matrix A with reduced row eschelon form R where

$$A = \begin{bmatrix} \frac{5}{2} & \frac{4}{3} & -\frac{5}{2} & 0 & -\frac{2}{3} \\ -\frac{5}{4} & -\frac{2}{3} & \frac{5}{4} & \frac{4}{3} & -\frac{7}{6} \\ -\frac{15}{2} & -4 & \frac{15}{2} & \frac{4}{9} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad R = \begin{array}{c} \begin{matrix} P & F & F & P & P \end{matrix} \\ \begin{bmatrix} 1 & \frac{8}{15} & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

(i) Find a basis for Col(A) $= \{ Ax : x \in \mathbb{R}^5 \}$

$$\left\{ \begin{bmatrix} 5/2 \\ -5/4 \\ -15/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/3 \\ 4/9 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -7/6 \\ 1/2 \end{bmatrix} \right\}$$

solve for x_1

(ii) Find a basis for Nul(A) $= \{ x : Ax = 0 \} = \{ x : Rx = 0 \}$

$$\left\{ \begin{aligned} & x_1 + \tfrac{8}{15} x_2 - x_3 = 0 \\ & x_4 = 0 \\ & x_5 = 0 \end{aligned} \right.$$

$$x = \begin{bmatrix} -\frac{8}{15} x_2 + x_3 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8/15 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3$$

basis

Basis is

$$\left\{ \begin{bmatrix} -8/15 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

New

5. Let A be the matrix and x be the vector given by

$$A = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Show that x is an eigenvector of A and find the eigenvalue.

find $Ax = \ldots$

6. Answer the following true false questions:

(i) Whenever a system has free variables, the solution set contains a unique solution.

(A) True

(B) False

(ii) An inconsistent system has more than one solution.

(A) True

(B) False

(iii) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

(A) True

(B) False

(iv) $\det(A^{-1}) = 1/\det(A)$.

(A) True

(B) False

(v) Cramer's rule can only be used for invertible matrices.

(A) True

(B) False

(vi) If W is a subspace of $\mathbf{R}^n$ and v is in both W and $W^\perp$, then $v = 0$.

(A) True

(B) False

(vii) If $A = QR$ where Q has orthonormal columns, then $R = Q^T A$.

(A) True

(B) False

New

(viii) If $A \in \mathbf{R}^{n\times n}$ is symmetric, there exists an orthonormal basis of $\mathbf{R}^n$ which consists of eigenvectors of A.

Spectral theorem

(A) True

(B) False

(ix) Every matrix $A \in \mathbf{R}^{n\times n}$ can be factored as $A = SDS^{-1}$ where D is diagonal and S is an invertible matrix.

(A) True

(B) False

7. Suppose $A \in \mathbf{R}^{2\times 3}$ is given by

Exam 1

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 1 \end{bmatrix}.$$

How many free variables does the equation $Ax = 0$ have? Find all solutions to the equation $Ax = 0$.

8. Suppose $A \in \mathbf{R}^{2\times 2}$ is given by

exam 2

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to factor $A = QR$ where Q is a matrix with orthonormal columns and R is upper triangular.

9. Find the eigenvalues and eigenvectors of the matrix A where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

New

10. The LU factorization of a matrix A is given by

Exam

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/3 & -2/3 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Explain how to use this factorization to solve the equation $Ax = b$ and then find the value of x corresponding to $b = (4, 6, 17)$.

Exam 2

didn't change #'s

11. The QR factorization of a matrix A is given by

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{4}{3\sqrt{5}} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 3 & \frac{1}{3} \\ 0 & \frac{4\sqrt{5}}{3} \end{bmatrix}.$$

Explain how to use this factorization to minimize $\|Ax - b\|$ and then find the minimizing value of x corresponding to $b = (1, 0, 1)$.

12. The matrix A given by

New

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

has eigenvalues λ_i and eigenvectors x_i given by

$$\lambda_1 = 2, \quad x_1 = \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 3, \quad x_2 = \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}, \qquad \lambda_3 = 1, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

(i) What is D?

(ii) What is S?