Math 330: Quiz 1 Version C

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

I have worked independently on this quiz

— Sign your name here

2. Suppose $u, v \in \mathbf{R}^3$ and $A \in \mathbf{R}^{2\times 3}$ are given by

$$u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \qquad v = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix}.$$

(i) Find $2u - v$.

$$2\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 6+1 \\ -4-5 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ -9 \end{bmatrix}$$

(ii) Find Au.

$$\begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+6+2 \\ 3+0-4 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

Math 330: Quiz 1 Version C

I added these optional expanations to help you study for the final

3. Answer the following true false questions:

(i) Whenever a system has free variables, the solution set contains a unique solution.

(A) True

(B) False

Free variables indicates the nullspace is nontrivial and so any solution $Ax=b$ gives many more solutions of the form $x+y$ where $y \in \text{Nul}(A)$.

(ii) An inconsistent system has more than one solution.

(A) True

(B) False

Inconsistent means there are no solutions, though one could minimize $\|Ax-b\|$ to solve the least squares problem.

(iii) Every elementary row operation is reversible.

(A) True

(B) False

The reverse of $r_i \leftrightarrow r_j$ is $r_i \leftrightarrow r_j$
The reverse of $r_i \leftarrow r_i - \alpha r_j$ is $r_i \leftarrow r_i + \alpha r_j$ $(i \neq j)$
The reverse of $r_i \leftarrow \alpha r_i$ is $r_i \leftarrow \frac{1}{\alpha} r_i$ $(\alpha \neq 0)$

(iv) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

(A) True

(B) False

Composition of linear functions $f(x)=Ax$ and $g(x)=Bx$ is given by $(f \circ g)(x) = f(g(x)) = A(Bx) = ABx$ where AB is the matrix product for the resulting linear function.

4. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

Consider the linear system $Ax=b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Since there is a pivot in each row, then there are m pivots. Since each pivot must be in a different column that means there are m columns of A which contain a pivot. Those columns are linearly independent so they span $\mathbb{R}^m$. This means for any $b \in \mathbb{R}^m$ there is some linear combination of columns of A which equal b. Letting x be the coefficients in that linear combination yields $Ax=b$.

5. Let A be a 3×2 matrix. Explain why the equation $Ax = b$ cannot be consistent for all b in $\mathbf{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

Since $A \in \mathbb{R}^{3 \times 2}$ it has at most 2 pivots. This means $\text{Col}\,A$ has at most dimension 2. Since $\mathbb{R}^3$ is dimension 3, then there are many points $b \in \mathbb{R}^3$ that are not in $\text{Col}\,A$. The system $Ax=b$ can't be solved for such vectors b. If $A \in \mathbb{R}^{m \times n}$ where $n<m$, then $\dim \text{Col}\,A \leq n$. Since $n<m$ there are points $b \in \mathbb{R}^m$ that aren't in $\text{Col}\,A$. Again $Ax=b$ is inconsistent for those b's.

Math 330: Quiz 1 Version C

6. Write down the augmented matrix $\left[\, A \mid b \,\right]$ corresponding to the system of linear equations given by

$$\begin{cases} x_1 - 2x_2 + 7x_3 - 2x_4 = 8 \\ -x_1 + 7x_3 + 6x_4 = -3 \\ 2x_1 + 3x_2 + x_3 - 3x_4 = 5 \end{cases}$$

but *do not* solve these equations.

$$[A|b] = \begin{bmatrix} 1 & -2 & 7 & -2 & 8 \\ -1 & 0 & 7 & 6 & -3 \\ 2 & 3 & 1 & -3 & 5 \end{bmatrix}$$

7. Suppose $A \in \mathbf{R}^{2\times 3}$ is given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

How many free variables does the equation $Ax = 0$ have? Find all solutions to the equation $Ax = 0$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \quad r_2 \leftarrow r_2 - 2r_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \quad r_2 \leftarrow -1\, r_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad r_1 \leftarrow r_1 - 2r_2$$

$$R = \begin{array}{c} \begin{matrix} P & P & F \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

Thus there is one free variable

Solve $Ax = 0$ is the same as $Rx = 0$.

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

so

$$x_1 = x_3$$
$$x_2 = -2x_3$$

and all solutions are

$$x = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$ where x_3 is free.

Math 330: Quiz 1 Version C

8. The LU factorization of a matrix A is given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}.$$

Explain how to use this factorization to solve the equation $Ax = b$ and then find the value of x corresponding to $b = (0, -5, 7)$.

$Ax=b$ and $A=LU$ thus $LUx=b$. Setting $y=Ux$ yields two simpler systems of linear equations $\begin{cases} Ly=b \\ Ux=y \end{cases}$

Solve the first system for y and then the second for x to solve $Ax=b$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} \quad \text{so} \quad \begin{aligned} y_1 &= 0 \\ y_2 &= -5 - \tfrac{1}{2}y_1 \\ y_3 &= 7 - \tfrac{3}{2}y_1 + 5y_2 \end{aligned} \quad \text{so} \quad y = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$$

$$\begin{array}{r} 25 \\ -7 \\ \hline 18 \end{array}$$

Next

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix} \quad \text{so} \quad \begin{aligned} 2x_1 &= 0 + 2x_2 - 4x_3 \\ -2x_2 &= -5 + x_3 \\ -6x_3 &= -18 \end{aligned}$$

so
$$\begin{aligned} x_3 &= \frac{-18}{-6} = 3 \\ x_2 &= \frac{-5+x_3}{-2} = \frac{-5+3}{-2} = \frac{-2}{-2} = 1 \\ x_1 &= \frac{2x_2 - 4x_3}{2} = x_2 - 2x_3 = 1 - 6 = -5 \end{aligned} \quad \text{so} \quad x = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}.$$