Math 330: Exam 2 Version B

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

I have worked independently on the quiz

— Write your name here

2. Consider the matrix A with reduced row eschelon form R where

$$A = \begin{array}{c} \begin{array}{ccccc} P & F & F & P & P \end{array} \\ \begin{bmatrix} \frac{4}{3} & -1 & -3 & -3 & \frac{5}{2} \\ \frac{8}{3} & -2 & -6 & -\frac{7}{2} & \frac{17}{3} \\ \frac{16}{9} & -\frac{4}{3} & -4 & \frac{7}{2} & \frac{10}{3} \end{bmatrix} \end{array} \quad \text{and} \quad R = \begin{array}{c} \begin{array}{ccccc} P & F & F & P & P \end{array} \\ \begin{bmatrix} 1 & -\frac{3}{4} & -\frac{9}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(i) Find a basis for Col(A).

Basis: $\left\{ \begin{bmatrix} 4/3 \\ 8/3 \\ 16/9 \end{bmatrix}, \begin{bmatrix} -3 \\ -7/2 \\ 7/2 \end{bmatrix}, \begin{bmatrix} 5/2 \\ 17/3 \\ 10/3 \end{bmatrix} \right\}$

(ii) Find a basis for Nul(A). $Ax=0$ is the same as $Rx=0$

$$\begin{bmatrix} 1 & -\frac{3}{4} & -\frac{9}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0 \quad \text{so} \quad \begin{aligned} x_1 &= \tfrac{3}{4}x_2 + \tfrac{9}{4}x_3 \\ \left.\begin{aligned} x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}\right\} &\text{ free} \\ x_4 &= 0 \\ x_5 &= 0 \end{aligned} \quad \text{so} \quad x = \begin{bmatrix} 3/4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 9/4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3$$

Basis: $\left\{ \begin{bmatrix} 3/4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

I added these explanations to help you study for the final exam.

3. Answer the following true false questions:

(i) $\det(A^{-1}) = (-1)\det(A)$.

Since $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$ then $\det(A^{-1}) = \frac{1}{\det(A)}$

(A) True

(B) False

(ii) Cramer's rule can only be used for invertible matrices.

(A) True

(B) False

The formula involved dividing by $\det A$. That means $\det A \neq 0$ which implies A is invertible.

(iii) If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $x \to Ax$ preserves lengths.

Orthonormal columns implies $A^T A = I$. Thus $\|Ax\|^2 = Ax \cdot Ax = (Ax)^T Ax = x^T A^T A x = x^T I x = x \cdot x = \|x\|^2$ implies $\|Ax\| = \|x\|$.

(A) True

(B) False

(iv) If $A = QR$ where Q has orthonormal columns, then $R = Q^T A$.

Since $Q^T Q = I$ then $Q^T A = Q^T Q R = IR = R$

(A) True

(B) False

4. Let U be a square matrix such that $U^T U = I$. Show that $\det(U) = \pm 1$.

By properties of determinants $\det(U^T)\det(U) = \det(U^T U) = \det I = 1$ and $\det(U^T) = \det(U)$.

Therefore $(\det U)^2 = 1$. This implies $\det U = \pm 1$.

5. What is the rank of a 4×5 matrix whose null space is two dimensional?

Let $A \in \mathbb{R}^{4\times 5}$ then $\operatorname{rank} A = \dim \operatorname{col} A = \#$ of pivot variables.

Since $\dim \operatorname{Nul} A = 2 = \#$ of free variables and the total $\#$ of variables is 5. Then

$$\operatorname{rank} A + 2 = 5$$

implies $\operatorname{rank} A = 3$.

6. Find det(A), det(B) and det(AB) where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad r_1 \leftrightarrow r_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det A = \det \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix} = 3 \cdot 4 \cdot 5 = 60$$

$$\det B = \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$$

$$\det AB = (\det A)(\det B) = (60)(-1) = -60$$

7. Suppose $A \in \mathbf{R}^{2\times 2}$ is given by

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to factor $A = QR$ where Q is a matrix with orthonormal columns and R is upper triangular.

$$t_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$t_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{5}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{2/4}} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{1/\sqrt{2}} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and
$$R = \begin{bmatrix} \sqrt{8} & 5/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

8. The QR factorization of a matrix A is given by

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{4}{3\sqrt{5}} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 3 & \frac{1}{3} \\ 0 & \frac{4\sqrt{5}}{3} \end{bmatrix}.$$

Explain how to use this factorization to minimize $\|Ax - b\|$ and then find the minimizing value of x corresponding to $b = (1, 0, 1)$.

To minimize $\|Ax-b\|$ solve $Rx = Q^Tb$.

$$Q^Tb = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ \frac{2}{3\sqrt{5}} & \sqrt{5}/3 & 4/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 + 2/3 \\ \frac{2}{3\sqrt{5}} + \frac{4}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Thus $Rx = Q^Tb$ implies

$$\begin{bmatrix} 3 & 1/3 \\ 0 & \frac{4\sqrt{5}}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{so} \quad \begin{aligned} 3x_1 &= 1 - \tfrac{1}{3}x_2 \\ \tfrac{4\sqrt{5}}{3}x_2 &= \tfrac{2}{\sqrt{5}} \end{aligned}$$

$$x_2 = \frac{2 \cdot 3}{4 \cdot 5} = \frac{3}{10}$$

so $$x_1 = \frac{1 - \frac{1}{3}x_2}{3} = \frac{1 - \frac{1}{10}}{3} = \frac{9}{10} \cdot \frac{1}{3} = \frac{3}{10}$$

Therefore the answer is $x = \begin{bmatrix} 3/10 \\ 3/10 \end{bmatrix}$.