The unknowns

important...

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

$A \in \mathbb{R}^{n\times n}$

constraints

$\underset{n\times n}{A}\,\underset{n}{x} = \lambda \underset{n}{x}$ } This is n equations written in vector form...

Solve for $x^{\in \mathbb{R}^n}$ and λ

n unknowns one more

Total of $n+1$ unknowns

degrees of freedom

Note: one extra degree of freedom, ..

What's that extra degree of freedom?

Suppose x and λ satisfy

$$Ax = \lambda x$$

Take $y = 4x$ then what?

$$Ay = A(4x) = 4Ax = 4\lambda x = \lambda 4x = \lambda y$$

Thus $Ay = \lambda y$

and so y and λ are also solutions...

The extra degree of freedom is that the length of the eigenvector x is flexible...
i.e. the length of x isn't determined...

In some applications we'll solve this indeterminacy by choosing x to be a unit vector...

Solve $Ax = \lambda x$

$$Ax - \lambda x = 0$$

$$\boxed{(\underbrace{A - \lambda I}_{A-\lambda I \in \mathbb{R}^{n\times n}})x = 0}$$

for $x \neq 0$ to be a solution it means $A-\lambda I$ must have free variables...

actually says $x \in \text{Nul}(A - \lambda I)$

Conditions for $x \neq 0$ to exist

- $A - \lambda I$ has free variables
- $\text{Nul}(A-\lambda I)$ contains more than just the zero vector...
- $A - \lambda I$ is not invertible
- $\boxed{\det(A - \lambda I) = 0}$

This can be viewed as an equation for λ that doesn't involve x..

impractical idea... use this equation and the definition of determinant to solve for λ. Then plug those values of λ into $\mathrm{Nul}(A-\lambda I)$ to find x...

Note the are iterative methods that resemble the way Newton's method works for solving $f(x)=0$ that can be used to solve the eigenvalue-eigenvector problem $Ax=\lambda x$.

Example... find eigenvalues and eigenvector when

$$\text{et } A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \mathbf{u} =$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda)-(5)(6)$$

$$= \lambda^2 - 3\lambda + 2 - 30 = \lambda^2 - 3\lambda - 28 = 0$$

rational root theorem to factor

$$= (\lambda-7)(\lambda+4) = 0$$

so $\lambda = 7$ or $\lambda = -4$

λ. Then

$\mathrm{Nul}(A-\lambda I)$

Case $\lambda = 7$: $x \in \mathrm{Nul}(A-7I)$

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7I = \boxed{\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}} \rightarrow$

$-6x_1 + 6x_2 = 0$

$x_1 = x_2$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$

Here is an eigenvector, and for definiteness take $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus we have $\boxed{(\lambda, x) = \left(7, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}$ is an eigenvalue eigenvector pair for the matrix A.

Case $\lambda = -4$: $x \in \mathrm{Nul}(A+4I)$

(note this part was finished after class)

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - (-4)I = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$

$5x_1 + 6x_2 = 0$

$x_1 = -\frac{6}{5} x_2$

$x = \begin{bmatrix} -6/5 \\ 1 \end{bmatrix} x_2$

For definiteness take $x_2 = 5$ so the eigenvector $x = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$.

Thus $\boxed{(\lambda, x) = \left(-4, \begin{bmatrix} -6 \\ 5 \end{bmatrix}\right)}$ is another eigenvalue eigenvector pair for A.