Observation: If A is upper triangular then the eigenvalues appear on the diagonal...

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Claim the eigenvalues are 1, 4, 6

Trying to solve $Ax = \lambda x$ for a $x \neq 0$ by care choosing λ so this possible..

Choose λ so that $\det(A - \lambda I) = 0$ or equivalently so $\mathrm{Nul}(A - \lambda I)$ is non-trivial.

$$\mathrm{Nul}(A - \lambda I) \neq \{0\}.$$

Choose λ so $A - \lambda I$ has free variables

Check the claim

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Claim the eigenvalues are 1, 4, 6

$\underline{\underline{\lambda=1}}$ $$A - I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

thus $\mathrm{Nul}(A-I)$ is non-trivial.

free variable here

alternatively it's easy to see

$$\det \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} = 0$$

$\underline{\lambda=4}$ $$A - 4I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

free vbl.

$\underline{\lambda=6}$ exactly the same ...

Note if $A = LU$ where U is the row eschelon form of A found by Gaussian elimination — unfortunately there is no direct relationship between the eigenvalues of A and the eigenvalues of U. Note there is some connection but it's not enough to map eigen values back and forth

Similarity transformation: Idea — keep the eigenvalues the same, but change the matrix...

$A \in \mathbb{R}^{n \times n}$ Let $S \in \mathbb{R}^{n \times n}$ such that S^{-1} exists.

Let $B = SAS^{-1}$ we say B is similar to A.

Let λ be an eigenvalue of A. Thus there is an eigenvector x such that $Ax = \lambda x$.

Let $y = Sx$ then what

$$By = (SAS^{-1})y = SA\cancel{S^{-1}}\cancel{S}x = SAx = S\lambda x$$

since λ is a scalar

$$= \lambda Sx = \lambda y$$

Therefore $By = \lambda y$ shows λ is also an eigenvalue of B.

Consequently every eigenvalue of A is an eigenvalue of B.
Now I need to show the reverse... this can be obtained just by switching the roles of A and B since A is also similar to B see below

All I need to show is that

B is similar to A

implies

A is similar to B.

Now, B is similar to A means there is an invertible S such that $B = SAS^{-1}$

Need to show A is similar to B which means I need to find an inverible M such that $A = MBM^{-1}$.

Take $M = S^{-1}$ then

$$MBM^{-1} = S^{-1}B(S^{-1})^{-1} = S^{-1}BS$$

on the other hand $B = SAS^{-1}$

Therefore $MBM^{-1} = \cancel{S}^{-1}(\cancel{S}AS\cancel{^{-1}})\cancel{S} = A$.

(done).

The eigenvectors corresponding to different eigenvalues are linearly independent.

$$Ax_1 = \lambda_1 x_1 \qquad Ax_2 = \lambda_2 x_2$$

from the example $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

and $\lambda_1 = 7$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ also $\lambda_2 = -4$ $x_2 = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$