M 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall λ's are the solutions to $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 3 & 4 \\ 0 & 4-\lambda & -1 & 5 \\ 0 & 0 & 6-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \right)$$

$$= (2-\lambda)(4-\lambda)(6-\lambda)(1-\lambda) = 0$$

The solutions are $\lambda = 2$, $\lambda = 4$, $\lambda = 6$ and $\lambda = 1$.

Another way to see this

$$\text{Null}\left(\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 4I \right)$$

$$= \mathrm{Null}\left(\begin{array}{cccc} \begin{bmatrix} -2 & 1 & 3 & 4 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\ \begin{matrix} P & F & P & P \end{matrix} \end{array}\right)$$

3 pivots
1 free variable

Since there is a free variable then the nullspace is more than just the zero vector.
Any $x \in \mathrm{Null}(A - 4I)$ that not zero in an eigenvector with value 4, and there are plenty of them.

REM 4 If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Definition the matrix $B \in \mathbb{R}^{n\times n}$ is said to be similar to $A \in \mathbb{R}^{n \times n}$ if there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that

$$\boxed{B = SAS^{-1}}$$

Claim if B is similar to A then A is similar to B

Explain why? any ideas?

Need to find an invertible matrix $M \in \mathbb{R}^{n \times n}$ such that $A = MBM^{-1}$.

Just set $M = S^{-1}$ and check.

$$MBM^{-1} = S^{-1}B(S^{-1})^{-1} = S^{-1}BS = S^{-1}(SAS^{-1})S$$

$$= \cancel{S^{-1}S}\, A\, \cancel{S^{-1}S} = A$$

Suppose λ and x are eigenvalue/eigenvector pair for A.

Thus $\boxed{Ax = \lambda x}$ ☑

$B = SAS^{-1}$ Let $y = Sx$

$$By = SAS^{-1}\underbrace{y}_{x} = SA\cancel{S^{-1}S}x = SAx$$

$$= S\lambda x = \lambda Sx = \lambda y$$

Thus $By = \lambda y$ ...