The eigenvectors corresponding to different eigenvalues are linearly independent.

Why?

Suppose $\boxed{\lambda_1, \lambda_2, \ldots \lambda_p}$ are eigenvalues with corresponding eigenvectors $\boxed{x_1, x_2, \ldots, x_p}$ and $\lambda_i \neq \lambda_j$ whenever $i \neq j$.

means $Ax_1 = \lambda_1 x_1$
$Ax_2 = \lambda_2 x_2$
$\vdots$
$Ax_p = \lambda_p x_p$

Case $p=4$. That is 4 vectors and 4 eigenvalues.

Suppose

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0$$

to show the x_i's are independent, I need to show the only solution is when $c_1 = c_2 = c_3 = c_4 = 0$.

$$A(c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4) = A0$$

$$c_1 A x_1 + c_2 A x_2 + c_3 A x_3 + c_4 A x_4 = 0$$

idea eliminate c_1

$$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + c_3 \lambda_3 x_3 + c_4 \lambda_4 x_4 = 0$$
$$c_1 \lambda_1 x_1 + c_2 \lambda_1 x_2 + c_3 \lambda_1 x_3 + c_4 \lambda_1 x_4 = 0$$

mult. the original equation by λ_1 and subtract...

$$c_2(\lambda_2 - \lambda_1) x_2 + c_3(\lambda_3 - \lambda_1) x_3 + c_4(\lambda_4 - \lambda_1) x_4 = 0$$

mult by A and then subtract to eliminate the x_2 term

get this from mult by A

subtract

$$c_2(\lambda_2 - \lambda_1)\lambda_2 x_2 + c_3(\lambda_3 - \lambda_1)\lambda_3 x_3 + c_4(\lambda_4 - \lambda_1)\lambda_4 x_4 = 0$$
$$c_2(\lambda_2 - \lambda_1)\lambda_2 x_2 + c_3(\lambda_3 - \lambda_1)\lambda_2 x_3 + c_4(\lambda_4 - \lambda_1)\lambda_2 x_4 = 0$$

$$c_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) x_3 + c_4(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2) x_4 = 0$$

from mult by A

subtract

$$\begin{cases} \cancel{c_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\lambda_3 x_3 + c_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)\lambda_4 x_4 = 0 \\ c_3(\lambda_3-\lambda_1)\cancel{(\lambda_3-\lambda_2)}\lambda_3 x_3 + c_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)\lambda_3 x_4 = 0 \end{cases}$$

mult by A and then subtract to eliminate the x_3 term

$$c_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)x_4 = 0$$

a set of one non-zero vector is always a linearly ind. set.

Since $x_4 \neq 0$ This means

$$c_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3) = 0$$

since the eigenvalues are all different none of these terms are zero.

Thus $c_4 = 0$.

Since $c_4 = 0$ plug it into

$$c_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)x_3 + \cancel{c_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)x_4} = 0$$

so $c_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)x_3 = 0$

Thus $c_3 = 0$

Since $c_3 = 0$ and $c_4 = 0$ plug it into

$$c_2(\lambda_2-\lambda_1)x_2 + c_3\cancel{(\lambda_3-\lambda_1)}x_3 + c_4\cancel{(\lambda_4-\lambda_1)}x_4 = 0$$

So $c_2(\lambda_2-\lambda_1)x_2 = 0$

Thus $c_2 = 0$

Since $c_2 = c_3 = c_4 = 0$ plug it in to

$$c_1 x_1 + \cancel{c_2 x_2} + \cancel{c_3 x_3} + \cancel{c_4 x_4} = 0$$

So $c_1 x_1 = 0$

Thus $c_1 = 0$.

This implies $c_1 = c_2 = c_3 = c_4 = 0$ so the x_i's are linearly independent.