ITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$\det \begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 4 & 2 & 1 \\ 5 & 0 & -2 & 8 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

For each row choose a different column

example...

There are $4! = 24$ different way to do this..

To get the determinant multiply the choices in each permutation together and add them up alternating the sign..

$\pm 2 \cdot 3 \cdot 8 + 1$ + 23 other similar terms...

Back to eigenvectors and eigenvalues...

$Ax = \lambda x$ and try to solve for λ and x.

$(A - \lambda I)x = 0$

eliminate the x using determinants to obtain

$\det(A - \lambda I) = 0$ ← solve this for λ and then find the corresponding x's by plugging those values of λ in...

In general $A \in \mathbb{R}^{n \times n}$ and one of the term in $\det(A - \lambda I)$ is the product of the diagonals

Note there are $n!$ total terms... At any rate we see $\det(A - \lambda I)$ is a polynomial of degree n.

Do we?

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 4 & 2 & 1 \\ 5 & 0 & -2 & 8 \\ 1 & 2 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 2 & 3 & 6 \\ 3 & 4-\lambda & 2 & 1 \\ 5 & 0 & -2-\lambda & 8 \\ 1 & 2 & 1 & 2-\lambda \end{bmatrix}\right)$$

each of these term involve at least one element that off the diagonal...

$$= \underbrace{(1-\lambda)(4-\lambda)(-2-\lambda)(2-\lambda)}_{\text{poly of degree 4}} + \underbrace{23 \text{ other things}}_{\text{largest this poly could be is degree 3...}}$$

doesn't cancel the λ^4 term over here

$$= \text{poly of degree } 4$$

Fundamental theorem of Algebra: Every polynomial of degree n has n roots (possibly complex) counted by multiplicity.

That is, every polynomal of degree n can be factored into n factors using complex numbers.

$\det(A-\lambda I)$ in general has n roots λ counted by multiplicity.

from last week:

rvation: the eigenvectors corresponding to different eigenvalues are linearly independent.

If the roots of $\det(A-\lambda I)$ are all different then the above observation means

$Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, ... , $Ax_n = \lambda_n x_n$

and $x_1, x_2, \dots, x_n$ are all linearly independent...

Since any n linearly independent vectors in $\mathbb{R}^n$ form a basis then $\{x_1, x_2, \dots, x_n\}$ is a basis

$$\underbrace{\{x_1, x_2, \dots, x_n\}}_{\text{eigenbasis...}}$$

- If all the roots of $\det(A - \lambda I)$ are different, then it's guarenteed you have an eigenbasis.
- If same are repeated, there may or may not be an eigenbasis...

The eigenvector turns matrix-vector multiplication onto scalar multiplication which is much simpler...

Why is an eigenbasis important?

$x \in \mathbb{R}$ then $x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

$Ax = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n$

I've turned Ax into a sum of scalar multiplications...

Example when things go wrong

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3$$

eigenvalues are $\lambda = 3$ with mult. 3.

All the eigenvectors for $\lambda = 3$ are given as non-zero solutions to $Ax = 3x$..

These are the non-zero elements of Nul $(A-3I)$

$$(A-3I)x = \begin{bmatrix} \overset{F}{0} & \overset{P}{1} & \overset{P}{0} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

$x_2 = 0$
$x_3 = 0$

all solutions are of the form

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$$

Only one linearly independent vector in Nul$(A-3I)$
so there isn't an eigenbasis in this case...