From last time...

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalues and eigenvectors from last time were

① $\lambda_1 = 2$ and $x_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

② $\lambda_2 = -1$ and $x = \begin{bmatrix} 5/3 \\ 1 \\ 0 \end{bmatrix}$ rescale to make fraction go away $x_2 = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$

③ $\lambda_3 = 4$ and $x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Since eigenvectors corresponding to different eigenvalues are linearly independent (Theorem ?) then

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an eigenbasis...

- Use the eigenbasis to "diagonalize" the matrix A by means of a similarity transformation

Thus, I want a diagonal matrix D and and invertible matrix S such that

$$A = \underbrace{SDS^{-1}}_{\text{similarity transformation by } S\ldots}$$

this is a factorization based on eigenvalues and eigenvectors...

This means A is similar to D and we know the eigenvalues of two similar matrices are the same. Consequently D has the eigenvalues of A on it's diagonal.

Let S be the matrix whose columns are the eigenbasis corresponding to A. That is the n linear eigen vectors of A.

Let

$$S = \left[\, x_1 \,\middle|\, x_2 \,\middle|\, x_3 \,\right] = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 3 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{r|r} -1 & -3 \\ 0 & 0 \\ 3 & 3 \\ \hline 2 & 0 \end{array}$$

Then

$$AS = A\left[\, x_1 \,\middle|\, x_2 \,\middle|\, x_3 \,\right] = \left[\, Ax_1 \,\middle|\, Ax_2 \,\middle|\, Ax_3 \,\right]$$

it's easier to multiply the vector by the scalar

$$Ax_1 = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \text{ and } \lambda_1 x_1 = 2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

these are the same

our eigenvalue eigenvector calculation from last time was correct...

$$AS = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{D}$$

the diagonal matrix with eigenvalues on the diagonal

Therefore

$$AS = SD$$

$$ASS^{-1} = SDS^{-1}$$

$$\boxed{A = SDS^{-1}}$$

factorization of A using the eigenbasis...

similarity of A with D.

what is this factorization useful for?

$$A^2 = AA = (SDS^{-1})(SDS^{-1}) = SD\cancel{(S^{-1}S)}DS^{-1} = SD^2S^{-1}$$

squaring A is the same as squaring D inside the similarity transformation...

In fact

$$A^k = SD^kS^{-1} \quad \text{for } k = 1, 2, 3, \ldots$$

much easier to compute

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D^2 = DD = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

Next Idea: $p(t) = 4t^2 - 3t + 9$

Then $p(A) = \underbrace{4A^2 - 3A + 9I}_{\text{simplify this using } A = SDS^{-1}}$