Next Idea: $p(t) = 4t^2 - 3t + 9$

Then $P(A) = \underbrace{4A^2 - 3A + 9I}$

simplify this using $A = SDS^{-1}$

assume $A \in \mathbb{R}^{n \times n}$ that has an eigenbasis $\{x_1, x_2, \ldots, x_n\}$

(If A does not have an eigenbasis, it's still possible to simplify, but more difficult and saved for Math 430.

Therefore, we have $A = SDS^{-1}$ where D is a diagonal matrix of eigenvalues..

$$P(A) = 4A^2 - 3A + 9I = 4(SDS^{-1})^2 - 3SDS^{-1} + 9I$$

Note

$$(SDS^{-1})^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1}$$

$$P(A) = 4SD^2S^{-1} - 3SDS^{-1} + 9SS^{-1}$$

$$= S(4D^2S^{-1} - 3DS^{-1} + 9S^{-1})$$

$$= S(\underbrace{4D^2 - 3D + 9I}_{P(D)})S^{-1}$$

Thus

$$\boxed{\underbrace{P(A)}_{SDS^{-1}} = SP(D)S^{-1}}$$

In general for any B and any invertible matrix S we have

$$P(SBS^{-1}) = SP(B)S^{-1}$$

Now let's use the fact that D is diagonal ...

$$P(D) = 4D^2 - 3D + 9I \quad \text{where } D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Important thing about diagonal matrices...

$$D^2 = D \cdot D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix}$$

Note for non-diagonal matrices

$$\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}^2 \neq \begin{bmatrix} (-1)^2 & 0 & 1^2 \\ (-3)^2 & 4^2 & 1^2 \\ 0 & 0 & 2^2 \end{bmatrix}$$

completely wrong...

$$P(D) = 4D^2 - 3D + 9I$$

$$= 4\begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix} - 3\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} + 9\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\lambda_1^2 & & & 0 \\ & 4\lambda_2^2 & & \\ & & \ddots & \\ 0 & & & 4\lambda_n^2 \end{bmatrix} + \begin{bmatrix} -3\lambda_1 & & & 0 \\ & -3\lambda_2 & & \\ & & \ddots & \\ 0 & & & -3\lambda_n \end{bmatrix} + \begin{bmatrix} 9 & & & 0 \\ & 9 & & \\ & & \ddots & \\ 0 & & & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4\lambda_1^2 - 3\lambda_1 + 9 & & & 0 \\ & 4\lambda_2^2 - 3\lambda_2 + 9 & & \\ & & \ddots & \\ 0 & & & 4\lambda_n^2 - 3\lambda_n + 9 \end{bmatrix}$$

$$= \begin{bmatrix} P(\lambda_1) & & & 0 \\ & P(\lambda_2) & & \\ & & \ddots & \\ 0 & & & P(\lambda_n) \end{bmatrix}$$

In summary

$$P(A) = S\,P(D)\,S^{-1} = S\begin{bmatrix} P(\lambda_1) & & & 0 \\ & P(\lambda_2) & & \\ & & \ddots & \\ 0 & & & P(\lambda_n) \end{bmatrix}S^{-1}$$

What about functions other than polynomials?

For any function with a convergent power series you can do this same thing.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \frac{1}{5!}A^5 + \cdots$$

If A has an eigenbasis this power series can be simplifies the same way as for a polynomial...

$$e^A = S \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix} S^{-1}$$

Try to remember power series for sine and cosine for next time...