Spectral in linear algebra means about the eigenvalues and consequently the eigenvectors ...

- Spectral Theorem: If $A \in \mathbb{R}^{n \times n}$ is symmetric then A has an orthonormal basis of eigenvectors.

Underlying concept: since orthonormal basis are so usefull anything that can be done in a application to obtain a symmetric matrix instead of one that isn't is worth doing.

So were solving $Ax = \lambda x$

① $\underbrace{\det(A - \lambda I)}_{\text{nth degree polynomial ...}} = 0$

by the Fundamental theorem of Algebra this equation has n solutions counted by multiplicity and they might be complex...

If $A = A^T$ then the eigenvalues $\lambda \in \mathbb{R}$. Why?

Note if λ is complex then substituting it into $\text{Nul}(A - \lambda I)$ and solving for x likely gives complex eigenvectors as well...

Note that even if x is complex then $\|x\|$ is real.

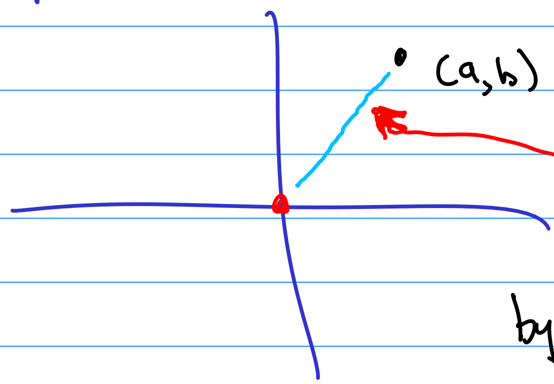
$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

$$\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

$$= x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n$$

what is a complex number $a+ib$

Complex plane



$(a,b) = a+ib$

$|a+ib|$ is the distance to the origin

by Pythagorean theorem

$$|a+ib| = \sqrt{a^2+b^2}$$

or again

$$|a+ib|^2 = a^2+b^2$$

What's the complex conjugate?

$$\overline{a+ib} = a-ib$$

For example

$$\overline{3+2i} = 3-2i$$

What happens

$$(3+2i)(3-2i) = 9 - 6i + 6i - (2i)^2 = 9+4$$

Similarly

$$(a+ib)(a-ib) = a^2+b^2 = |a+ib|^2$$

Therefore

$$\|x\|^2 = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$$

$$= x_1 \bar{x}_1 + x_2 \bar{x}_2 + \dots + x_n \bar{x}_n$$

$$\|x\|^2 = x \cdot \bar{x}$$

$$Ax \cdot \bar{x} = \lambda x \cdot \bar{x} = \lambda \|x\|^2$$

$$\|$$

$$(Ax)^T \bar{x} = \underbrace{x^T A^T \bar{x} = x^T A \bar{x}}_{\text{Symmetry}} = x^T \bar{A} \bar{x}$$

the fact that $A \in \mathbb{R}^{n \times n}$

$$= x^T \overline{Ax} = x^T \overline{\lambda x} = \bar{\lambda} x^T x = \bar{\lambda} \|x\|^2$$

Therefore

$$\lambda \|x\|^2 = \bar{\lambda} \|x\|^2$$

This means $\lambda = \bar{\lambda}$ or that λ is real...